C. D. Chester
31 May 2021

$$\int_0^{\infty} (x^n+1)^{-1}\ dx = \frac{\pi/n}{\sin(\pi/n)}$$

Acknowledgment

I have always wanted to prove this generalized integral multiple ways. Before this I proved a couple cases (where $n = 0, 1, 2, 3$ with 0 and 1 being common integrals) and with the Beta Function for general purposes. However, using Complex Analysis feels far more satisfying. I wanted to specifically credit Dr. Peyam and Crystal Math for giving video tools that I could extrapolate! See the References for those videos. AS for integral since I don't know how to draw it I'll be using a sector contour with the angle defined below.

Proof

$$\therefore \oint_C f(z)\ dz = \int_0^R f(z)\ dz + \int_{\gamma_R} f(z)\ dz - \underbrace{e^{\frac{2\pi i}{n}}}_{\theta=\frac{2\pi}{n}} \int_0^R f(z)\ dz \tag{1}$$

$$= \left(1 - e^{\frac{2\pi i}{n}}\right) \int_0^R f(z)\ dz + \int_{\gamma_R} f(z)\ dz \tag{2}$$

$$\therefore \text{Res}[f(z)] \Longrightarrow \lim_{z\to e^{\frac{\pi i}{n}}} \left(z - e^{\frac{\pi i}{n}}\right)(z^n+1)^{-1} \tag{3}$$

$$= \lim_{z\to e^{\frac{\pi i}{n}}} \frac{1}{nz^{n-1}} = \frac{1}{ne^{\pi i(1-1/n)}} \tag{4}$$

$$\therefore \oint_C f(z)\ dz = \frac{2\pi i}{ne^{\pi i(1-1/n)}} \tag{5}$$

$$\therefore \left|\int_{\gamma_R} f(z)\ dz\right| \leq \left(\frac{2\pi R}{n}\right)(R^n-1)^{-1} \approx \frac{1}{R^{n-1}} \Longrightarrow \lim_{R\to\infty}\left(\frac{1}{R^{n-1}}\right) = 0 \tag{6}$$

$$\therefore \frac{2\pi i}{ne^{\pi i(1-1/n)}} = \left(1 - e^{\frac{2\pi i}{n}}\right) \lim_{R\to\infty}\left[\int_0^R f(z)\ dz\right] \tag{7}$$

$$\therefore \int_0^{\infty} f(z)\ dz = \frac{2\pi i}{ne^{\pi i(1-1/n)}\left(1 - e^{\frac{2\pi i}{n}}\right)} \tag{8}$$

$$= \frac{2\pi i}{n(e^{i\pi(1-1/n)} - e^{i\pi(1+1/n)})} = \frac{2\pi i}{n(e^{i\pi(1-1/n)} - e^{-i\pi(1-1/n)})} \tag{9}$$

$$= \frac{2\pi i}{n2i\sin(\pi(1-1/n))} = \frac{\pi/n}{\sin(\pi(1-1/n))} = \frac{\pi/n}{\sin(\pi/n)} \tag{10}$$

References

[1] **C. D. Chester.** "Integral of $1/(x^n + 1)$ from 0 to ∞" *C. D. Publications*, 9 Apr 19, `https://cdchester.co.uk/2019/04/09/integral-of-1-xn1-from-0-to-%e2%88%9e/`. Accessed 31 May 21.

[2] **C. D. Chester.** "Integral of $[1 + (x^g)]^{-n}]$ from 0 to ∞" *C. D. Publications*, 17 Apr 19, `https://cdchester.co.uk/2019/04/17/integral-of-1xg-n-from-0-to-%e2%88%9e/`. Accessed 31 May 21.

[3] **Crystal Math.** "Contour Integration #7 - Integrating $1/(x^8 + 1)$ - LearnMathsFree" *YouTube*, 8 Aug 16, `https://youtu.be/Qy-TNya9GE0`. Accessed 31 May 21.

[4] **Dr Peyam.** "Integral $1/(x^n + 1)$ from 0 to infinity" *YouTube*, 23 Mar 18, `https://youtu.be/xro7c-mDk1g`. Accessed 31 May 21.