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$$\int_0^\infty \arctan^3(x^{-1})\ dx$$

1: u-Substitution & DI Method #1

$$\therefore\ I = \int_0^\infty \arctan^3(x^{-1})\ dx \tag{1}$$

$$\therefore\ x = \frac{1}{u} \Longrightarrow dx = -\frac{du}{u^2}\ ;\ |_{0\Longrightarrow\infty}^{\infty\Longrightarrow 0} \tag{2}$$

$$\therefore\ I = \int_0^\infty \arctan^3(u)\cdot u^{-2}\ du \tag{3}$$

$$\therefore\ \frac{d}{du}\left(\arctan^3(u)\right) = 3\arctan^2(u)(u^2+1)^{-1} \tag{4}$$

$$\therefore\ \int u^{-2}\ du = -u^{-1} + C_1 \tag{5}$$

$$\therefore\ I = (-u^{-1})\arctan^3(u)|_0^\infty + \int_0^\infty 3\arctan^2(u)\left(u^{-1}\right)(u^2+1)^{-1}\ du \tag{6}$$

$$\therefore\ R_1 = (-u^{-1})\arctan^3(u)|_0^\infty = (x)\arctan^3(x^{-1})|_0^\infty \tag{7}$$

$$\therefore\ I_1 = \int_0^\infty \arctan^2(u)\left(u^{-1}\right)(u^2+1)^{-1}\ du \tag{8}$$

$$\therefore\ I = R_1 + 3I_1 \tag{9}$$

2: g-Substitution & DI Method #2

$$\therefore\ I_1 = \int_0^\infty \arctan^2(u)u^{-1}(u^2+1)^{-1}\ du \tag{10}$$

$$\therefore\ g = \arctan(u) \Longrightarrow dg = \frac{du}{u^2+1}\ ;\ |_{0\Longrightarrow 0}^{\infty\Longrightarrow\pi/2} \tag{11}$$

$$\therefore\ I_1 = \int_0^{\pi/2} g^2\cot(g)\ dg \tag{12}$$

$$\therefore\ \frac{d}{dg}\left(g^2\right) = 2g \tag{13}$$

$$\therefore\ \int \cot(g)\ dg = \ln|\sin(g)| + C_2 \tag{14}$$

$$\therefore\ I_1 = g^2\ln|\sin(g)||_0^{\pi/2} - \int_0^{\pi/2} 2g\ln|\sin(g)|\ dg \tag{15}$$

$$\therefore\ R_2 = g^2\ln|\sin(g)||_0^{\pi/2} \tag{16}$$

$$= \arctan^2(u)\ln|\sin(\arctan(u))||_0^\infty \tag{17}$$

$$= -\arctan^2(x^{-1})\ln|\sin(\arctan(x^{-1}))||_0^\infty \tag{18}$$

$$\therefore\ I_2 = \int_0^{\pi/2} g\ln|\sin(g)|\ dg \tag{19}$$

$$\therefore\ I_1 = R_2 - 2I_2 \tag{20}$$

$$\therefore\ I = R_1 + 3R_2 - 6I_2 \tag{21}$$

3: Euler Formula & DI Method #3

$$\therefore\ I_2 = \int_0^{\pi/2} g \ln|\sin(g)|\ dg \tag{22}$$

$$\therefore\ \sin(g) = \left(-\frac{i}{2}\right)\left(e^{ig} - e^{-ig}\right) \tag{23}$$

$$\therefore\ I_2 = \int_0^{\pi/2} g \ln\left|\left(-\frac{i}{2}\right)\left(e^{ig} - e^{-ig}\right)\right|\ dg \tag{24}$$

$$\therefore\ \frac{d}{dg}\left(\ln|\sin(g)|\right) = \cot(g) = \frac{0.5(e^{ig}+e^{-ig})}{-0.5i(e^{ig}-e^{-ig})} = i\left(\frac{e^{2ig}+1}{e^{2ig}-1}\right) \tag{25}$$

$$\therefore\ \int g\ dg = 0.5g^2 + C_3 \tag{26}$$

$$\therefore\ I_2 = 0.5g^2 \ln|\sin(g)||_0^{\pi/2} - \int_0^{\pi/2}\left(\frac{i}{2}\right)(g^2)\left(\frac{e^{2ig}+1}{e^{2ig}-1}\right)\ dg \tag{27}$$

$$= 0.5R_2 - \int_0^{\pi/2}\left(\frac{i}{2}\right)(g^2)\left(\frac{e^{2ig}+1}{e^{2ig}-1}\right)\ dg \tag{28}$$

$$\therefore\ I_3 = \int_0^{\pi/2} g^2\left(\frac{e^{2ig}+1}{e^{2ig}-1}\right)\ dg \tag{29}$$

$$\therefore\ I_2 = \left(\frac{1}{2}\right)R_2 - \left(\frac{i}{2}\right)I_3 \tag{30}$$

$$\therefore\ I = R_1 + 3R_2 - 6\left(\left(\frac{1}{2}\right)R_2 - \left(\frac{i}{2}\right)I_3\right) = R_1 + (3i)I_3 \tag{31}$$

4: n-Substitution & Partial Fraction Decomposition

$$\therefore\ I_3 = \int_0^{\pi/2} g^2\left(\frac{e^{2ig}+1}{e^{2ig}-1}\right)\ dg \tag{32}$$

$$\therefore\ n = e^{2ig} - 1 \implies dn = (2i)e^{2ig}\ dg \implies dg = \left(-\frac{i}{2}\right)e^{-2ig}\ dn\ ;\ |_{0\implies 0}^{\pi/2\implies -2} \tag{33}$$

$$\therefore\ g^2 = -\frac{\ln^2|n+1|}{4} \tag{34}$$

$$\therefore\ I_3 = \int_{-2}^{0} -\left(\frac{i}{2}\right)\left(\frac{\ln^2|n+1|}{4}\right)\left(\frac{n+2}{n(n+1)}\right)\ dn = \int_{-2}^{0} -\left(\frac{i}{8}\right)\ln^2|n+1|\left(\frac{n+2}{n(n+1)}\right)\ dn \tag{35}$$

$$\therefore\ \frac{n+2}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1} \implies A = 2\ ;\ B = -1 \tag{36}$$

$$\therefore\ I_3 = \int_{-2}^{0} -\left(\frac{i}{8}\right)\ln^2|n+1|\left(\frac{2}{n} - \frac{1}{n+1}\right)\ dn \tag{37}$$

$$\therefore\ I_4 = \int_{-2}^{0} \ln^2|n+1|\left(\frac{2}{n} - \frac{1}{n+1}\right)\ dn \tag{38}$$

$$\therefore\ I = R_1 - (3i)\left(\frac{i}{8}\right)I_4 = R_1 + \left(\frac{3}{8}\right)I_4 \tag{39}$$

$$\therefore\ I_4 = \int_{-2}^{0} \ln^2|n+1| \left(\frac{2}{n} - \frac{1}{n+1}\right)\ dn \tag{40}$$

$$\therefore\ I_5 = \int_{-2}^{0} \ln^2|n+1| \left(\frac{1}{n}\right)\ dn\ ;\ I_6 = \int_{-2}^{0} \ln^2|n+1| \left(\frac{1}{n+1}\right)\ dn \tag{41}$$

$$\therefore\ I_4 = 2I_5 - I_6 \tag{42}$$

$$\therefore\ I = R_1 + \left(\frac{3}{8}\right)(2I_5 - I_6) \tag{43}$$

$$\therefore\ I_5 = \int_{-2}^{0} \ln^2|n+1| \left(\frac{1}{n}\right)\ dn \tag{44}$$

$$\therefore\ \frac{d}{dn}\left(\ln^2|n+1|\right) = \frac{2\ln|n+1|}{n+1} \tag{45}$$

$$\therefore\ \int \frac{1}{n}\ dn = \ln|-n| + C_4 \tag{46}$$

$$\therefore\ I_5 = \ln|-n|\ln^2|n+1||_{-2}^{0} - \int_{-2}^{0} \frac{2\ln|-n|\ln|n+1|}{n+1}\ dn \tag{47}$$

$$\therefore\ R_3 = \ln|-n|\ln^2|n+1||_{-2}^{0}\ ;\ I_7 = \int_{-2}^{0} \frac{\ln|-n|\ln|n+1|}{n+1}\ dn \tag{48}$$

$$\therefore\ I = R_1 + \left(\frac{3}{8}\right)(2(R_3 - 2I_7) - I_6) = R_1 + \left(\frac{3}{4}\right)R_3 - \left(\frac{3}{8}\right)(I_6 + 4I_7) \tag{49}$$

$$\therefore\ I_6 = \int_{-2}^{0} \ln^2|n+1| \left(\frac{1}{n+1}\right)\ dn = \frac{\ln^3|n+1|}{3}|_{-2}^{0} = 0 \tag{50}$$

$$\therefore\ I_7 = \int_{-2}^{0} \frac{\ln|-n|\ln|n+1|}{n+1}\ dn \tag{51}$$

$$\therefore\ \frac{d}{dn}\left(\ln|n+1|\right) = \frac{1}{n+1} \tag{52}$$

$$\therefore\ \int \frac{\ln|-n|}{n+1}\ dn = \mathrm{Li}_2(n+1) + C_5 \tag{53}$$

$$\therefore\ I_7 = \ln|n+1|\mathrm{Li}_2(n+1)|_{-2}^{0} - \int_{-2}^{0} \frac{\mathrm{Li}_2(n+1)}{n+1}\ dn \tag{54}$$

$$= \ln|n+1|\mathrm{Li}_2(n+1) - \mathrm{Li}_3(n+1)|_{-2}^{0} \tag{55}$$

$$= -\mathrm{Li}_3(1) + \mathrm{Li}_3(-1) = \left(\frac{7}{4}\right)\zeta(3) \tag{56}$$

$$\therefore\ R_1 = 0,\ R_3 = \pi^2\ln(2) \tag{57}$$

$$\therefore\ I = \left(\frac{3}{4}\right)\pi^2\ln(2) - \left(\frac{21}{8}\right)\zeta(3) \approx 1.97542 \tag{58}$$