π and $\ln(x)$ Studies Vol. 1

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Outline

1 Personal Approximation of π Using WRA

1.1 General Form

Using WRA I created a formula in the general form of:

$$\pi \approx \ln(\alpha) + \frac{1}{\beta} + \frac{1}{\gamma} - \frac{1}{\kappa} + \frac{1}{\lambda} \tag{1}$$

$$\alpha = 23 \tag{2}$$

$$\beta = 164 \tag{3}$$

$$\gamma = 1140660 \tag{4}$$

$$\kappa = 2538436614635 \tag{5}$$

$$\lambda = 18288891096136284325165309 \tag{6}$$

1.2 Tackling the Logarithm

The only hard part, besides the tedious fractional computations alone, is the ln(23) portion. Here's what I did to challenge it.

$$\ln\left(\frac{24}{23}\right) = \sum_{k=1}^{\infty} \frac{1}{a^k \cdot k}; a = 24 \tag{7}$$

$$\ln(24) - \ln(23) = \tag{8}$$

$$\ln(23) = \ln(24) - \sum_{k=1}^{\infty} \frac{1}{24^k \cdot k} \tag{9}$$

$$= 3\ln(2) + \ln(3) - \sum_{k=1}^{\infty} \frac{1}{24^k \cdot k} \tag{10}$$

$$= \sum_{k=1}^{\infty} \frac{4}{2^k \cdot k} + \frac{1}{3^k \cdot k} - \frac{1}{24^k \cdot k} \tag{11}$$

$$= \sum_{k=1}^{\infty} \frac{1}{k}\left(\frac{4}{2^k} + \frac{1}{3^k} - \frac{1}{24^k}\right) \tag{12}$$

$$= 3.1354942159291496908067528318101 \ldots \tag{13}$$

1.3 Results

The approximation yielded by my selected values gives a raw error of just over 2.25 x 10^{-52} and a percentage error from π of roughly 7.17 x 10^{-51}%.

2 Natural Logarithm Series

2.1 $\ln\left|\dfrac{x}{x-1}\right|$ Series Representation

$$\ln\left|\frac{x}{x-1}\right| = \sum_{k=1}^{\infty} \frac{1}{k \cdot x^k} \tag{14}$$

2.2 $\ln(2)$

$$\ln(2) = \sum_{k=1}^{\infty} \frac{1}{k \cdot 2^k} \tag{15}$$

2.3 $\ln(3)$

$$\ln(3) = \ln\left(\frac{3}{2}\right) + \ln(2) \tag{16}$$

$$= \sum_{k=1}^{\infty} \frac{1}{k \cdot 3^k} + \sum_{k=1}^{\infty} \frac{1}{k \cdot 2^k} \tag{17}$$

$$= \sum_{k=1}^{\infty} \frac{1}{k}\left(\frac{1}{2^k} + \frac{1}{3^k}\right) \tag{18}$$

$$= 2 \cdot \ln(2) - \ln\left(\frac{4}{3}\right) \tag{19}$$

$$= 2\sum_{k=1}^{\infty} \frac{1}{k \cdot 2^k} - \sum_{k=1}^{\infty} \frac{1}{k \cdot 4^k} \tag{20}$$

$$= \sum_{k=1}^{\infty} \frac{1}{k}\left(\frac{2}{2^k} - \frac{1}{4^k}\right) \tag{21}$$

$$= \frac{1}{2}\left(\ln\left(\frac{9}{8}\right) + 3\ln(2)\right) \tag{22}$$

$$= \frac{1}{2}\left(\sum_{k=1}^{\infty} \frac{1}{k \cdot 9^k} + 3\sum_{k=1}^{\infty} \frac{1}{k \cdot 2^k}\right) \tag{23}$$

$$= \frac{1}{2}\sum_{k=1}^{\infty} \left(\frac{1}{k}\right)\left(\frac{3}{2^k} + \frac{1}{9^k}\right) \tag{24}$$

2.4 $\ln(5)$

$$\ln(5) = \ln\left(\frac{5}{4}\right) + 2\ln(2) \tag{25}$$
$$= \sum_{k=1}^{\infty} \frac{1}{k \cdot 5^k} + 2 \cdot \sum_{k=1}^{\infty} \frac{1}{k \cdot 2^k} \tag{26}$$
$$= \sum_{k=1}^{\infty} \left(\frac{1}{k}\right)\left(\frac{2}{2^k} + \frac{1}{5^k}\right) \tag{27}$$

$$= \frac{1}{2}\left(\ln\left(\frac{25}{24}\right) + 3\ln(2) + \ln(3)\right) \tag{28}$$
$$= \frac{1}{2}\left(\sum_{k=1}^{\infty} \frac{1}{k \cdot 25^k} + 3\sum_{k=1}^{\infty} \frac{1}{k \cdot 2^k} + \sum_{k=1}^{\infty} \frac{1}{k}\left(\frac{1}{2^k} + \frac{1}{3^k}\right)\right) \tag{29}$$
$$= \frac{1}{2}\sum_{k=1}^{\infty} \frac{1}{k}\left(\frac{4}{2^k} + \frac{1}{3^k} + \frac{1}{25^k}\right) \tag{30}$$

2.5 $\ln(6)$

$$\ln(6) = \ln(2) + \ln(3) \tag{31}$$
$$= \sum_{k=1}^{\infty} \frac{1}{k \cdot 2^k} + \sum_{k=1}^{\infty} \frac{1}{k}\left(\frac{2}{2^k} - \frac{1}{4^k}\right) \tag{32}$$
$$= \sum_{k=1}^{\infty} \frac{1}{k}\left(\frac{3}{2^k} - \frac{1}{4^k}\right) \tag{33}$$

$$= \frac{1}{2}\left(\ln\left(\frac{36}{35}\right) + \ln(5) + \ln(7)\right) \tag{34}$$
$$= \frac{1}{2}\left(\sum_{k=1}^{\infty} \frac{1}{k \cdot 36^k} + \sum_{k=1}^{\infty} \left(\frac{1}{k}\right)\left(\frac{2}{2^k} + \frac{1}{5^k}\right) + \sum_{k=1}^{\infty} \frac{1}{k}\left(\frac{3}{2^k} - \frac{1}{4^k} + \frac{1}{7^k}\right)\right) \tag{35}$$
$$= \frac{1}{2}\sum_{k=1}^{\infty} \frac{1}{k}\left(\frac{5}{2^k} - \frac{1}{4^k} + \frac{1}{5^k} + \frac{1}{7^k} + \frac{1}{36^k}\right) \tag{36}$$

2.6 $\ln(7)$

$$\ln(7) = \ln\left(\frac{7}{6}\right) + \ln(2) + \ln(3) \tag{37}$$

$$= \sum_{k=1}^{\infty} \frac{1}{k \cdot 7^k} + \sum_{k=1}^{\infty} \frac{1}{k \cdot 2^k} + \sum_{k=1}^{\infty} \frac{1}{k}\left(\frac{2}{2^k} - \frac{1}{4^k}\right) \tag{38}$$

$$= \sum_{k=1}^{\infty} \frac{1}{k}\left(\frac{3}{2^k} - \frac{1}{4^k} + \frac{1}{7^k}\right) \tag{39}$$

$$= \frac{1}{2}\left(\ln\left(\frac{49}{48}\right) + 4\ln(2) + \ln(3)\right) \tag{40}$$

$$= \frac{1}{2}\left(\sum_{k=1}^{\infty} \frac{1}{k \cdot 49^k} + \sum_{k=1}^{\infty} \frac{4}{k \cdot 2^k} + \sum_{k=1}^{\infty} \frac{1}{k}\left(\frac{2}{2^k} - \frac{1}{4^k}\right)\right) \tag{41}$$

$$= \frac{1}{2}\sum_{k=1}^{\infty} \frac{1}{k}\left(\frac{6}{2^k} - \frac{1}{4^k} + \frac{1}{49^k}\right) \tag{42}$$

2.7 $\ln(10)$

$$\ln(10) = \ln(2) + \ln(5) \tag{43}$$

$$= \sum_{k=1}^{\infty} \frac{1}{k \cdot 2^k} + \sum_{k=1}^{\infty} \left(\frac{1}{k}\right)\left(\frac{2}{2^k} + \frac{1}{5^k}\right) \tag{44}$$

$$= \sum_{k=1}^{\infty} \frac{1}{k}\left(\frac{3}{2^k} + \frac{1}{5^k}\right) \tag{45}$$

$$= \ln\left(\frac{10}{9}\right) + 2\ln(3) \tag{46}$$

$$= \sum_{k=1}^{\infty} \frac{1}{k \cdot 10^k} + 2\sum_{k=1}^{\infty} \frac{1}{k}\left(\frac{1}{2^k} + \frac{1}{3^k}\right) \tag{47}$$

$$= \sum_{k=1}^{\infty} \left(\frac{1}{k}\right)\left(\frac{2}{2^k} + \frac{2}{3^k} + \frac{1}{10^k}\right) \tag{48}$$

2.8 $\ln(11)$

$$\ln(11) = 2 \cdot \ln(2) + \ln(3) - \ln\left(\frac{12}{11}\right) \tag{49}$$

$$= \sum_{k=1}^{\infty} \frac{2}{k \cdot 2^k} + \sum_{k=1}^{\infty} \frac{1}{k}\left(\frac{2}{2^k} - \frac{1}{4^k}\right) - \sum_{k=1}^{\infty} \frac{1}{k \cdot 12^k} \tag{50}$$

$$= \sum_{k=1}^{\infty} \frac{1}{k}\left(\frac{4}{2^k} - \frac{1}{4^k} - \frac{1}{12^k}\right) \tag{51}$$

$$= \ln\left(\frac{11}{10}\right) + \ln(10) \tag{52}$$

$$= \sum_{k=1}^{\infty} \frac{1}{k \cdot 11^k} + \sum_{k=1}^{\infty} \frac{1}{k}\left(\frac{3}{2^k} + \frac{1}{5^k}\right) \tag{53}$$

$$= \sum_{k=1}^{\infty} \frac{1}{k}\left(\frac{3}{2^k} + \frac{1}{5^k} + \frac{1}{11^k}\right) \tag{54}$$

3 Alternating and Non-Alternating BBP Series, $n = 2$

3.1 Alternating Series

$$\beta_A(x, n, (a_1, a_2, \ldots, a_n)) = \sum_{k=0}^{\infty} (-1)^k \cdot x^{nk} \cdot \left(\frac{a_1}{nk+1} + \ldots + \frac{a_n}{nk+n} \right) \tag{55}$$

$$= \int_0^1 \frac{\sum_{k=1}^{n} (a_k \cdot u^{k-1})}{1 + (ux)^n} \, du \; ; \; x \in (-1, 1] \tag{56}$$

$$\therefore \; n = 2 \Longrightarrow \int_0^1 \frac{\sum_{k=1}^{2} (a_k \cdot u^{k-1})}{1 + (ux)^2} \, du \tag{57}$$

$$= \frac{a_1}{x} \cdot \arctan(x) + \frac{a_2}{2x^2} \cdot \ln |x^2 + 1| \tag{58}$$

$$\therefore \; a_1 = 0 \Longrightarrow \frac{a_2}{2x^2} \cdot \ln |x^2 + 1| \tag{59}$$

$$\therefore \; x^2 + 1 = \frac{1}{m} \Longrightarrow |x| = \sqrt{\frac{1}{m} - 1} \leq 1 \Longrightarrow 1 \leq 2m \leq 2 \tag{60}$$

$$\therefore \; \beta_A(\sqrt{\frac{1}{m} - 1}, 2, (0, a_2)) = \frac{a_2 \ln(m)}{2\left(1 - \frac{1}{m}\right)} \; ; \; 1 \leq 2m \leq 2 \tag{61}$$

$$\therefore \; \frac{a_2 \ln(2)}{2} \leq \beta_A(x, 2, (0, a_2)) \leq \frac{a_2}{2} \tag{62}$$

$$\therefore \; a_2 = 0 \Longrightarrow \frac{a_1}{x} \cdot \arctan(x) \tag{63}$$

$$\therefore \; \beta_A(x, 2, (a_1, 0)) = \frac{a_1}{x} \cdot \arctan(x) \; ; \; 0 \leq x \leq 1 \tag{64}$$

$$\therefore \; \frac{a_1 \pi}{4} \leq \beta_A(x, 2, (a_1, 0)) \leq a_1 \tag{65}$$

$$\therefore \; \lim_{x \to 0} \beta_A(x, 2, (a_1, a_2)) = a_1 + \frac{a_2}{2} \tag{66}$$

3.2 Non-Alternating Series

$$\beta(x,n,(a_1,a_2,\dots,a_n)) = \sum_{k=0}^{\infty} x^{nk} \cdot \left(\frac{a_1}{nk+1} + \dots + \frac{a_n}{nk+n}\right) \quad (67)$$

$$= \int_0^1 \frac{\sum_{k=1}^{n}(a_k \cdot u^{k-1})}{1-(ux)^n}\, du \ ; \ x \in (-1,1) \quad (68)$$

$$\therefore\ n = 2 \Longrightarrow \int_0^1 \frac{\sum_{k=1}^{2}(a_k \cdot u^{k-1})}{1-(ux)^2}\, du \quad (69)$$

$$= \frac{a_1}{x} \cdot \tanh^{-1}(x) - \frac{a_2}{2x^2} \cdot \ln|x^2-1| \quad (70)$$

$$\therefore\ a_1 = 0 \Longrightarrow -\frac{a_2}{2x^2} \cdot \ln|x^2-1| \quad (71)$$

$$\therefore\ x^2 - 1 = -c \ ; \ -1 < -c < 0 \quad (72)$$

$$\therefore\ \beta\left(\sqrt{1-c}, 2, (0,a_2)\right) = \frac{a_2 \ln|c|}{2\,(c-1)} \ ; 0 < c < 1 \quad (73)$$

$$\therefore\ \frac{a_2}{2} < \beta(x,2,(0,a_2)) < \infty \quad (74)$$

$$\therefore\ a_2 = 0 \Longrightarrow \frac{a_1}{x} \cdot \tanh^{-1}(x) \quad (75)$$

$$\therefore\ \beta(x,2,(a_1,0)) = \frac{a_1}{x} \cdot \tanh^{-1}(x) = \frac{a_1}{2x} \cdot \ln\left|\frac{1+x}{1-x}\right| \ ; \ |x| < 1 \quad (76)$$

$$\therefore\ x = \frac{\alpha-1}{\alpha+1} \ ; \ \alpha \in (0,\infty) \Longrightarrow \frac{a_1}{2} \cdot \frac{\alpha+1}{\alpha-1} \cdot \ln(\alpha) \quad (77)$$

$$\therefore\ a_1 \leq \beta(x,2,(a_1,0)) < \infty \quad (78)$$

$$\therefore\ \lim_{x\to 0} \beta(x,2,(a_1,a_2)) = a_1 + \frac{a_2}{2} \quad (79)$$

4 General Logarithmic BBP Formulas

4.1 $\beta(x, 2, (a_1, 0))$

As I just showed in Section 3.2, by letting $a_2 = 0$ and $x = \dfrac{\alpha - 1}{\alpha + 1}$, you can create a simple expression for any positively argumented logarithm multiplied by a constant relative to α and a_1. Let's fix it to get specifically just the logarithm part.

$$\therefore\ \beta(x, 2, (a_1, 0)) = \frac{a_1}{x} \cdot \tanh^{-1}(x) = \frac{a_1}{2x} \cdot \ln\left|\frac{1+x}{1-x}\right| \ ;\ |x| < 1 \tag{80}$$

$$\therefore\ x = \frac{\alpha - 1}{\alpha + 1} \ ;\ \alpha \in (0, \infty) \Longrightarrow \frac{a_1}{2} \cdot \frac{\alpha + 1}{\alpha - 1} \cdot \ln(\alpha) \tag{81}$$

$$\therefore\ \beta\left(\frac{\alpha - 1}{\alpha + 1}, 2, (2, 0)\right) = \frac{\alpha + 1}{\alpha - 1} \cdot \ln(\alpha) \tag{82}$$

$$\therefore\ \frac{\alpha - 1}{\alpha + 1} \cdot \beta\left(\frac{\alpha - 1}{\alpha + 1}, 2, (2, 0)\right) = \ln(\alpha) \tag{83}$$

$$= \sum_{k=0}^{\infty} \left(\frac{\alpha - 1}{\alpha + 1}\right)^{2k+1} \left(\frac{2}{2k+1}\right) \tag{84}$$

4.1.1 $\alpha = 2$

Using $\alpha = 2$ yields a classic series representation for $\ln(2)$ (it is even represented on WRA here: `https://mathworld.wolfram.com/BBP-TypeFormula.html`.

$$\ln(2) = \sum_{k=0}^{\infty} \left(\frac{2-1}{2+1}\right)^{2k+1} \left(\frac{2}{2k+1}\right) \tag{85}$$

$$= \frac{2}{3} \sum_{k=0}^{\infty} \left(\frac{1}{9}\right)^{k} \left(\frac{1}{2k+1}\right) \tag{86}$$

This simple series of $\ln(2) = \displaystyle\sum_{k=1}^{\infty} \frac{1}{k \cdot 2^k}$ yields a digit for every 3.3 terms approximately. The above BBP series yields a digit for roughly every 1.1 terms (3x faster results).

4.1.2 $\alpha = 3$

Using $\alpha = 3$ yields a classic series representation for ln(3) (it is even represented in "A Compendium of BBP-Type Formulas for Mathematical Constants" by Bailey here on page 5 under Equation 6: `https://www.davidhbailey.com/dhbpapers/bbp-formulas.pdf`).

$$\ln(3) = \sum_{k=0}^{\infty} \left(\frac{3-1}{3+1}\right)^{2k+1} \left(\frac{2}{2k+1}\right) \tag{87}$$

$$= \sum_{k=0}^{\infty} \left(\frac{1}{4}\right)^{k} \left(\frac{1}{2k+1}\right) \tag{88}$$

The semi-simple version of $\ln(3) = \sum_{k=1}^{\infty} \frac{1}{k}\left(\frac{2}{2^k} - \frac{1}{4^k}\right)$ yields a digit for approximately every 3.3 terms. The BBP series yielding a digit just shy of twice as quickly at roughly a digit every 1.65 terms. However, note that there are twice as many operations to consider in every iteration of the non-BBP series, thus it's actually almost four times as fast.

5 Solving Reciprocal Quartics

5.1 Cover-Up Method Madness

$$F(a,b,c,d) = \frac{1}{(x-a)(x-b)(x-c)(x-d)} \tag{89}$$

$$\therefore \frac{1}{(x-a)(x-b)(x-c)(x-d)} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c} + \frac{D}{x-d} \tag{90}$$

$$\therefore A = \frac{1}{(a-b)(a-c)(a-d)} \tag{91}$$

$$\therefore B = \frac{1}{(b-a)(b-c)(b-d)} \tag{92}$$

$$\therefore C = \frac{1}{(c-a)(c-b)(c-d)} \tag{93}$$

$$\therefore D = \frac{1}{(d-a)(d-b)(d-c)} \tag{94}$$

$$\tag{95}$$

5.2 Integrating $F(a,b,c,d)$

$$\int F(a,b,c,d)\ dx = A\ln|x-a| + B\ln|x-b| + C\ln|x-c| + D\ln|x-d| + \alpha \tag{96}$$

$$\therefore F(e^{\frac{i\pi}{4}}, e^{\frac{3i\pi}{4}}, e^{\frac{5i\pi}{4}}, e^{\frac{7i\pi}{4}}) = \frac{1}{x^4+1} \tag{97}$$

$$\therefore \lim_{\phi\to\infty^-} \int_0^{\phi} F(e^{\frac{i\pi}{4}}, e^{\frac{3i\pi}{4}}, e^{\frac{5i\pi}{4}}, e^{\frac{7i\pi}{4}})\ dx \tag{98}$$

$$= \lim_{\phi \to \infty^-} \left(A \ln \left| x - e^{\frac{i\pi}{4}} \right| + B \ln \left| x - e^{\frac{3i\pi}{4}} \right| + C \ln \left| x - e^{\frac{5i\pi}{4}} \right| + D \ln \left| x - e^{\frac{7i\pi}{4}} \right| \Big|_0^\phi \right) \quad (99)$$

$$= \lim_{\phi \to \infty^-} \left(A \ln \left| \phi - e^{\frac{i\pi}{4}} \right| + B \ln \left| \phi - e^{\frac{3i\pi}{4}} \right| + C \ln \left| \phi - e^{\frac{5i\pi}{4}} \right| + D \ln \left| \phi - e^{\frac{7i\pi}{4}} \right| \right) \quad (100)$$

$$- \left(A \ln \left| -e^{\frac{i\pi}{4}} \right| + B \ln \left| -e^{\frac{3i\pi}{4}} \right| + C \ln \left| -e^{\frac{5i\pi}{4}} \right| + D \ln \left| -e^{\frac{7i\pi}{4}} \right| \right) \quad (101)$$

$$= \lim_{\phi \to \infty^-} I_\phi - \left(\frac{Ai\pi}{4} + \frac{3Bi\pi}{4} + \frac{5Ci\pi}{4} + \frac{7Di\pi}{4} \right) \quad (102)$$

$$= \lim_{\phi \to \infty^-} I_\phi - \frac{i\pi}{4} \left(A + 3B + 5C + 7D \right) \quad (103)$$

$$\therefore\ A = \frac{1}{2^{1.5}(i-1)} = -\frac{i+1}{2^{2.5}} \quad (104)$$

$$\therefore\ B = \frac{1}{2^{1.5}(i+1)} = -\frac{i-1}{2^{2.5}} \quad (105)$$

$$\therefore\ C = \frac{1}{2^{1.5}(1-i)} = \frac{i+1}{2^{2.5}} \quad (106)$$

$$\therefore\ D = -\frac{1}{2^{1.5}(i+1)} = \frac{i-1}{2^{2.5}} \quad (107)$$

$$\implies \lim_{\phi \to \infty^-} I_\phi - \frac{i\pi}{4} \left(-\frac{i+1}{2^{2.5}} - 3\frac{i-1}{2^{2.5}} + 5\frac{i+1}{2^{2.5}} + 7\frac{i-1}{2^{2.5}} \right) \quad (108)$$

$$= \lim_{\phi \to \infty^-} I_\phi - \frac{i\pi}{4} \left(4\frac{i+1}{2^{2.5}} + 4\frac{i-1}{2^{2.5}} \right) \quad (109)$$

$$= \lim_{\phi \to \infty^-} I_\phi - i\pi \left(\frac{i+1}{2^{2.5}} + \frac{i-1}{2^{2.5}} \right) = \lim_{\phi \to \infty^-} I_\phi - i\pi \left(\frac{2i}{2^{2.5}} \right) \quad (110)$$

$$= \lim_{\phi \to \infty^-} I_\phi + \frac{\pi}{2^{1.5}} \quad (111)$$

It is fairly trivial to conceptually explain how $\lim_{\phi \to \infty^-} I_\phi = 0$. Note that $\lim_{x \to \infty^-} (x^4+1)^{-1} = 0$ and thus integrating 0 yields 0. This is not a rigorous explanation, but remembering how derivatives and integrals work conceptually simplifies the answer. Therefore, the end result is:

$$\lim_{\phi \to \infty^-} \int_0^\phi F(e^{\frac{i\pi}{4}}, e^{\frac{3i\pi}{4}}, e^{\frac{5i\pi}{4}}, e^{\frac{7i\pi}{4}})\, dx = \int_0^\infty \frac{1}{x^4+1}\, dx = \frac{\pi}{2^{1.5}} \quad (112)$$

6 Solving Reciprocal Sextics

6.1 Cover-Up Method Madness Part II

$$F(a,b,c,d,e,f) = \frac{1}{(x-a)(x-b)(x-c)(x-d)(x-e)(x-f)} \tag{113}$$
$$= \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c} + \frac{D}{x-d} + \frac{E}{x-e} + \frac{F}{x-f} \tag{114}$$
$$\therefore\ A = \frac{1}{(a-b)(a-c)(a-d)(a-e)(a-f)} \tag{115}$$
$$\therefore\ B = \frac{1}{(b-a)(b-c)(b-d)(b-e)(b-f)} \tag{116}$$
$$\therefore\ C = \frac{1}{(c-a)(c-b)(c-d)(c-e)(c-f)} \tag{117}$$
$$\therefore\ D = \frac{1}{(d-a)(d-b)(d-c)(d-e)(d-f)} \tag{118}$$
$$\therefore\ E = \frac{1}{(e-a)(e-b)(e-c)(e-d)(e-f)} \tag{119}$$
$$\therefore\ F = \frac{1}{(f-a)(f-b)(f-c)(f-d)(f-e)} \tag{120}$$

6.2 Integrating $F(a,b,c,d,e,f)$

$$\int F(a,b,c,d,e,f)\ dx = A\ln|x-a| + B\ln|x-b| + C\ln|x-c| + D\ln|x-d| \tag{121}$$
$$+ E\ln|x-e| + F\ln|x-f| + \alpha \tag{122}$$

$$\therefore\ F(e^{\frac{i\pi}{6}}, e^{\frac{3i\pi}{6}}, e^{\frac{5i\pi}{6}}, e^{\frac{7i\pi}{6}}, e^{\frac{9i\pi}{6}}, e^{\frac{11i\pi}{6}}) = \frac{1}{x^6+1} \tag{123}$$
$$\therefore\ \lim_{\phi\to\infty^-} \int_0^{\phi} F(e^{\frac{i\pi}{6}}, e^{\frac{3i\pi}{6}}, e^{\frac{5i\pi}{6}}, e^{\frac{7i\pi}{6}}, e^{\frac{9i\pi}{6}}, e^{\frac{11i\pi}{6}})\ dx \tag{124}$$

$$= \lim_{\phi \to \infty^-} I_\phi - \left(\frac{Ai\pi}{6} + \frac{3Bi\pi}{6} + \frac{5Ci\pi}{6} + \frac{7Di\pi}{6} + \frac{9Ei\pi}{6} + \frac{11Fi\pi}{6} \right) \tag{125}$$

$$= \lim_{\phi \to \infty^-} I_\phi - \frac{i\pi}{6} \left(A + 3B + 5C + 7D + 9E + 11F \right) \tag{126}$$

$$\therefore \; A = -\frac{i + \sqrt{3}}{12} \tag{127}$$

$$\therefore \; B = -\frac{i}{6} \tag{128}$$

$$\therefore \; C = -\frac{i - \sqrt{3}}{12} \tag{129}$$

$$\therefore \; D = \frac{i + \sqrt{3}}{12} \tag{130}$$

$$\therefore \; E = \frac{i}{6} \tag{131}$$

$$\therefore \; F = \frac{i - \sqrt{3}}{12} \tag{132}$$

$$\implies \lim_{\phi \to \infty^-} I_\phi - \frac{i\pi}{6} \left(6\frac{i + \sqrt{3}}{12} + 6\frac{i}{6} + 6\frac{i - \sqrt{3}}{12} \right) \tag{133}$$

$$= \lim_{\phi \to \infty^-} I_\phi - \frac{i\pi}{12} \left(i + \sqrt{3} + 2i + i - \sqrt{3} \right) \tag{134}$$

$$= \lim_{\phi \to \infty^-} I_\phi - \frac{i\pi}{12}(4i) \tag{135}$$

$$= \lim_{\phi \to \infty^-} I_\phi + \frac{\pi}{3} \tag{136}$$

It is fairly trivial to conceptually explain how $\lim_{\phi \to \infty^-} I_\phi = 0$. Note that $\lim_{x \to \infty^-} (x^6 + 1)^{-1} = 0$ and thus integrating 0 yields 0. This is not a rigorous explanation, but remembering how derivatives and integrals work conceptually simplifies the answer. Therefore, the end result is:

$$\lim_{\phi \to \infty^-} \int_0^\phi F(e^{\frac{i\pi}{6}}, e^{\frac{3i\pi}{6}}, e^{\frac{5i\pi}{6}}, e^{\frac{7i\pi}{6}}, e^{\frac{9i\pi}{6}}, e^{\frac{11i\pi}{6}}) \, dx = \int_0^\infty \frac{1}{x^6 + 1} \, dx = \frac{\pi}{3} \tag{137}$$