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$$\int_0^{\frac{\pi}{2}} \ln\left|(a\cos\theta)^2 + (b\sin\theta)^2\right| \; d\theta$$

1 - Solution

$$\therefore x = \tan\theta \implies d\theta = \frac{dx}{x^2+1} \tag{1}$$

$$\therefore \int_0^{\infty} \frac{\ln\left|\frac{a^2}{x^2+1} + \frac{(bx)^2}{x^2+1}\right|}{x^2+1}\,dx = \int_0^{\infty} \frac{\ln\left|\frac{a^2+(bx)^2}{x^2+1}\right|}{x^2+1}\,dx \tag{2}$$

$$= \int_0^{\infty} \frac{\ln\left|a^2+(bx)^2\right| - \ln\left|x^2+1\right|}{x^2+1}\,dx \tag{3}$$

$$= \underbrace{\int_0^{\infty} \frac{\ln\left|a^2+(bx)^2\right| - \ln\left|x^2+a^2\right|}{x^2+1}\,dx}_{I_1} + \underbrace{\int_0^{\infty} \frac{\ln\left|x^2+a^2\right| - \ln\left|x^2+1\right|}{x^2+1} + \; dx}_{I_2} \tag{4}$$

$$\therefore \; I = I_1 + I_2 \tag{5}$$

$$\therefore \; I_1 = \int_0^{\infty} \frac{\ln\left|a^2+(bx)^2\right| - \ln\left|x^2+a^2\right|}{x^2+1}\,dx = \int_0^{\infty} \frac{\left(\ln\left|a^2+(xy)^2\right| \,\Big|_{y=1}^{y=b}\right)}{x^2+1}\,dx \tag{6}$$

$$= \int_0^{\infty}\int_1^{b} \frac{2x^2y}{(x^2+1)(a^2+(xy)^2)}\,dy\,dx = \int_1^{b}\int_0^{\infty} \frac{2x^2y}{(x^2+1)(a^2+(xy)^2)}\,dx\,dy \tag{7}$$

$$= \int_1^{b}\int_0^{\infty} \left(\frac{2y}{y^2-a^2}\right)\left(\frac{1}{1+x^2} - \frac{a^2}{a^2+(xy)^2}\right)\,dx\,dy \tag{8}$$

$$= \int_1^{b} \left(\frac{2y}{y^2-a^2}\right)\left[\arctan(x) - \left(\frac{a}{y}\right)\arctan\left(\frac{xy}{a}\right)\right]_{x=0}^{x=\infty}\,dy \tag{9}$$

$$= \int_1^{b} \left(\frac{2y}{y^2-a^2}\right)\left[\frac{\pi}{2}\left(1-\frac{a}{y}\right)\right]\,dy = \pi\int_1^{b}\left(\frac{y}{y^2-a^2}\right)\left(1-\frac{a}{y}\right)\,dy \tag{10}$$

$$= \pi\int_1^{b} \frac{y-a}{y^2-a^2}\,dy = \pi\int_1^{b}\frac{1}{y+a}\,dy = \pi\ln|y+a|\,\Big|_{y=1}^{y=b} \tag{11}$$

$$= \pi\ln|a+b| - \pi\ln|a+1| \tag{12}$$

$$\therefore\ I_2 = \int_0^\infty \frac{\ln\left|x^2+a^2\right| - \ln\left|x^2+1\right|}{x^2+1}\ dx = \int_0^\infty \frac{\left(\ln|x^2+y^2|\ |_{y=1}^{y=a}\right)}{x^2+1}\ dx \tag{13}$$

$$= \int_0^\infty \int_1^a \frac{2y}{(x^2+1)(x^2+y^2)}\ dy\ dx = \int_1^a \int_0^\infty \frac{2y}{(x^2+1)(x^2+y^2)}\ dx\ dy \tag{14}$$

$$= \int_1^a \int_0^\infty \left(\frac{2y}{y^2-1}\right)\left(\frac{1}{1+x^2} - \frac{1}{x^2+y^2}\right)\ dx\ dy \tag{15}$$

$$= \int_1^a \left(\frac{2y}{y^2-1}\right)\left[\arctan(x) - \left(\frac{1}{y}\right)\arctan\left(\frac{x}{y}\right)\right]_{x=0}^{x=\infty}\ dy \tag{16}$$

$$= \int_1^a \left(\frac{2y}{y^2-1}\right)\left[\frac{\pi}{2}\left(1-\frac{1}{y}\right)\right]\ dy = \pi\int_1^b \left(\frac{y}{y^2-1}\right)\left(1-\frac{1}{y}\right)\ dy \tag{17}$$

$$= \pi\int_1^a \frac{y-1}{y^2-1}\ dy = \pi\int_1^a \frac{1}{y+1}\ dy = \pi\ln|y+1|\ |_{y=1}^{y=a} \tag{18}$$

$$= \pi\ln|a+1| - \pi\ln|1+1| = \pi\ln|a+1| - \pi\ln(2) \tag{19}$$

$$\therefore\ I = I_1 + I_2 \tag{20}$$

$$= (\pi\ln|a+b| - \pi\ln|a+1|) + (\pi\ln|a+1| - \pi\ln(2)) \tag{21}$$

$$= \pi\ln\left|\frac{a+b}{2}\right| \tag{22}$$

2 - Acknowledgment

This solution is due to Michael Penn, who posted it on his YouTube channel via [1]. I am simply writing it down as a normal math paper for further dissemination online. You can find his other videos via [2].

References

[1] **Penn, Michael.** "A Parametric Logarithmic Trigonometric Integral." *YouTube*, 16 August 2020, `https://youtu.be/NAWZx77ZOpw`. Accessed 8 December 2020.

[2] **Penn, Michael.** "Videos." *YouTube*, `https://www.youtube.com/c/MichaelPennMath/videos`. Accessed 9 December 2020.