$$I(a, b, c) = \int \frac{1}{(x-a)(x-b)(x-c)}\, dx$$

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Outline

Acknowledgments & Prelude

Steve Chow, known more so online as BlackPenRedPen [1] (abbreviated as BPRP), created a video on YouTube (YT) titled *The Cover-Up Method & Why It Works!* [2] back in 2017. It gave me an idea recently to apply this to a specific cubic integral that I had solved [3] before to see the results. The method applied is attributed to Oliver Heaviside (an English electrical engineer). [4] See the endnotes for several links to short papers written on this method.

$$\int_0^\infty \frac{1}{x^3+1}\,dx = \frac{2\pi}{3^{1.5}}$$

Generalized Result

Assume $a \neq b \neq c$.

$$I(a,b,c) = \int \frac{1}{(x-a)(x-b)(x-c)}\ dx \tag{1}$$

$$\therefore\ I(a,b,c) = \int \frac{\varphi_1}{x-a} + \frac{\varphi_2}{x-b} + \frac{\varphi_3}{x-c}\ dx \tag{2}$$

$$\therefore\ \varphi_1 = \frac{1}{(a-b)(a-c)} \tag{3}$$

$$\therefore\ \varphi_2 = \frac{1}{(b-a)(b-c)} \tag{4}$$

$$\therefore\ \varphi_3 = \frac{1}{(c-a)(c-b)} \tag{5}$$

$$\therefore\ I(a,b,c) = \frac{1}{(a-b)(a-c)} \ln|x-a| + \frac{1}{(b-a)(b-c)} \ln|x-b| + \frac{1}{(c-a)(c-b)} \ln|x-c| + C \tag{6}$$

Application to Specific Integral

$$I = \lim_{\phi \to \infty^-} \left(\int_0^{\phi} \frac{1}{x^3+1}\,dx \right) \tag{7}$$

$$\therefore\ x^3+1 = (x+1)(x^2-x+1) = (x+1)\left(x - \frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\left(x - \frac{1}{2} - \frac{\sqrt{3}}{2}i\right) \tag{8}$$

$$\therefore\ a = -1\ ,\ b = \frac{1}{2} - \frac{\sqrt{3}}{2}i\ ,\ c = \frac{1}{2} + \frac{\sqrt{3}}{2}i \tag{9}$$

$$\begin{aligned} \therefore\ \varphi_1 &= \frac{1}{(a-b)(a-c)} = \frac{1}{\left(-1 - \left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\right)\left(-1 - \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\right)} \\ &= \frac{2}{(-3+\sqrt{3}i)(-3-\sqrt{3}i)} = \frac{2}{(-3)^2 - (\sqrt{3}i)^2} = \frac{2}{9+3} = \frac{1}{6} \end{aligned} \tag{10}$$

$$\therefore\ \varphi_2 = \frac{1}{(b-a)(b-c)} = \frac{-2}{3+3\sqrt{3}i} \tag{11}$$

$$\therefore\ \varphi_3 = \frac{1}{(c-a)(c-b)} = \frac{-2}{3-3\sqrt{3}i} \tag{12}$$

$$\therefore\ I = \lim_{\phi \to \infty^-} \left(\frac{1}{6}\ln|x+1| - \frac{2}{3+3\sqrt{3}i}\ln\left|x - \frac{1}{2} + \frac{\sqrt{3}}{2}i\right| - \frac{2}{3-3\sqrt{3}i}\ln\left|x - \frac{1}{2} - \frac{\sqrt{3}}{2}i\right| \right)\Bigg|_{x=0}^{x=\phi} \tag{13}$$

$$= -\left(\frac{1}{6}\ln|1| - \frac{2}{3+3\sqrt{3}i}\ln\left|-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right| - \frac{2}{3-3\sqrt{3}i}\ln\left|-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right| \right) + \lim_{\phi \to \infty^-} I(\phi) \tag{14}$$

$$= -\left(-\frac{2}{3+3\sqrt{3}i}\left(\frac{2\pi}{3}i\right) - \frac{2}{3-3\sqrt{3}i}\left(\frac{-2\pi}{3}i\right) \right) + \lim_{\phi \to \infty^-} I(\phi) \tag{15}$$

$$= \frac{\pi}{3}\left(\frac{4i}{3+3\sqrt{3}i} - \frac{4i}{3-3\sqrt{3}i} \right) + \lim_{\phi \to \infty^-} I(\phi) = \frac{\pi}{3}\left(\frac{2}{\sqrt{3}}\right) + \lim_{\phi \to \infty^-} I(\phi) \tag{16}$$

$$= \lim_{\phi \to \infty^-} I(\phi) + \frac{2\pi}{3^{1.5}} \tag{17}$$

$$\therefore \lim_{\phi\to\infty^-} I(\phi) = \left(\frac{1}{6}\ln|x+1| - \frac{2}{3+3\sqrt{3}i}\ln\left|x-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right| - \frac{2}{3-3\sqrt{3}i}\ln\left|x-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right|\right)\Bigg|_{x=\phi} \tag{18}$$

$$= \frac{1}{6}\left(\ln|x+1| + (-1+\sqrt{3}i)\ln\left|x-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right| + (-1-\sqrt{3}i)\ln\left|x-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right|\right)\Bigg|_{x=\phi} \tag{19}$$

$$= \frac{1}{6}\left(\ln|x+1| + (\sqrt{3}i)\ln\left|\left(x+e^{\frac{2\pi i}{3}}\right)^2\right| + (-1-\sqrt{3}i)\ln\left|\left(x+e^{\frac{-2\pi i}{3}}\right)\left(x+e^{\frac{2\pi i}{3}}\right|\right)\right|_{x=\phi} \tag{20}$$

$$= \frac{1}{6}\left(\ln|x+1| + (\sqrt{3}i)\ln|Q_{2i}(x)| + (-1-\sqrt{3}i)\ln|Q_{2j}(x)|\right)\Bigg|_{x=\phi} \tag{21}$$

$$= \frac{1}{6}\left(\ln|Q_1(x)| + (\sqrt{3}i)\ln\left|\frac{Q_{2i}(x)}{Q_{2j}(x)}\right| - \ln|Q_{2j}(x)|\right)\Bigg|_{x=\phi} \tag{22}$$

$$= \frac{1}{6}\left(\ln|Q_1(x)| - \ln|Q_{2j}(x)|\right)\Bigg|_{x=\phi} \tag{23}$$

$$= iG(x)\Big|_{x=\phi} \implies \Re\left[\lim_{\phi\to\infty^-} I(\phi)\Big|_{x=\phi}\right] = 0 \tag{24}$$

$$\therefore I = \frac{2\pi}{3^{1.5}} \blacksquare \tag{25}$$

If this last part is confusing I showed that the remaining functions evaluated to specifically imaginary ones (in which I labeled (iG(x)). Since this integral has no imaginary part the limit must go to zero. This gladly leaves us with the original result I expected.

References

[1]BPRP's YouTube Channel
https://www.youtube.com/user/blackpenredpen
[2]*The Cover-Up Method & Why It Works!* by **BPRP**
https://www.youtube.com/watch?v=fgPviiv_oZs
[3]*Integral of 1/(x^3+1) from 0 to* ∞ by **C. D. Chester**
https://cdchester.co.uk/2019/04/05/integral-of-1-x31-from-0-to-%e2%88%9e/
[4]*Heaviside's Cover-up Method* by **Jeremy Orloff**
http://math.mit.edu/~jorloff/suppnotes/suppnotes03/h.pdf
Heaviside's Method by **Grant B. Gustafson**
http://www.math.utah.edu/~gustafso/HeavisideCoverup.pdf
Heavyside "Cover-up" Method for Partial Fractions
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http://web.mit.edu/jorloff/www/18.03-esg/notes/extra/pf-coverup.pdf