YouTube Integrals: Ep. 1-5

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Outline

Ep. 1 - Beta Function Applications

$$\int_0^\infty \left(1+x^{\,a}\right)^{-b}\ dx$$

Introduction

Many people will of have seen similar, probably simpler, versions of this in Calculus I to discover the arctan relation to $\int \left(1+x^2\right)^{-2}\ dx$. This integral aims to expand this initial relation we are taught through an exploration of factorials in the *Beta Function*[1]. Many times this function, *Hypergeometric Series*[2], and *Functional Transformations*[3] are integral (pun intended) to its solution. Sometimes it allows us to isolate a whole family of primitives[4] that turn out to be the same or off by a constant depending on the situation.

$$I(a,b) = \int_0^\infty \left(1+x^{\,a}\right)^{-b}\ dx \tag{1}$$

Proposed Solution

Depending on your preference you can use different notations, however, below I have listed the three forms I would use.

$$I(a,b) = \frac{\beta\left(b-\frac{1}{a},\frac{1}{a}\right)}{a} \tag{2a}$$

$$= \frac{\Gamma\left(b-\frac{1}{a}\right)!\Gamma\left(\frac{1}{a}\right)!}{a\Gamma\left(b\right)!} \tag{2b}$$

$$= \frac{\left(b-\frac{1}{a}-1\right)!\left(\frac{1}{a}-1\right)!}{a\left(b-1\right)!} \tag{2c}$$

Key Properties

Queen Property[5]

$$\int_0^\infty f(x)\,dx \Longrightarrow \int_0^\infty f\left(\frac{1}{x}\right) x^{-2}\,dx \tag{3}$$

A Particular Beta Function Integral Representation[6]

$$\beta(x,y) = \int_0^\infty (\phi)^{x-1}\,(1+\phi)^{-(x+y)}\;d\phi \tag{4}$$

Proof of Solution

$$I(a,b) = \int_0^\infty (1+x^{a})^{-b}\;dx \tag{5a}$$

$$\therefore (3) \Longrightarrow \int_0^\infty \left(1+x^{-a}\right)^{-b} x^{-2}\,dx \tag{5b}$$

$$= \int_0^\infty (x)^{ab-2}\,(1+x^{a})^{-b}\;dx \tag{5c}$$

$$\therefore u = x^{a} \Longrightarrow dx = \frac{(u)^{\frac{1}{a}-1}}{a}\,du \tag{5d}$$

$$\therefore (5d) \Longrightarrow \frac{1}{a}\int_0^\infty (u)^{b-\frac{1}{a}-1}(1+u)^{-b}\,du \tag{5e}$$

$$\therefore (4),(5e) \Longrightarrow x = b-\frac{1}{a},\; x+y=b \tag{5f}$$

$$\therefore (5f) \Longrightarrow x = b-\frac{1}{a},\; y=\frac{1}{a} \tag{5g}$$

$$\therefore (5g) \Longrightarrow I(a,b) = \frac{\beta\left(b-\frac{1}{a},\frac{1}{a}\right)}{a} \tag{5h}$$

$$= (2a)\ \blacksquare \tag{5i}$$

Ep. 2 - Generalized Dirichlet Integral

$$\int_0^\infty \frac{sin(x)}{x^a}\,dx$$

Introduction

I was first introduced to this integral family by BPRP[7], when he made a video using *Differentiation Under the Integral Sign*[8] (Leibniz Integral Rule or Feynman's Trick are different names for this technique). I believe he called the Dirichlet Integral the "Main Dish" as a food joke.[9] For context, note that BPRP had made a few videos using food puns and this was his grand finale. Below is a slight modification of the Dirichlet Integral[10] through adding the a variable.

$$I(a) = \int_0^\infty \frac{sin(x)}{x^a}\,dx \tag{6}$$

Proposed Solution

The solution of this can be easily reached through a nifty *Laplace Transform Integral Property.*[11] I have yet to see a name for this property other than a mention on Wikipedia and MathWorld as a method of evaluating integrals over the positive, real axis. From now on I'll call this use of Laplace (and Inverse Laplace) Transforms under these types of integrals as the *Functional Laplace Transformation Property.* If that name is already taken send me an email or comment on one of my YouTube videos.

$$I(a) = \frac{\pi}{2(a-1)!}\csc\left(\frac{a\pi}{2}\right) \tag{7}$$

Key Properties

Functional Laplace Transformation Property[12]

$$\int_0^\infty f(x)g(x)\,dx = \int_0^\infty \mathscr{L}\left(f(x)\right)\mathscr{L}^{-1}\left(f(x)\right)\,dx \tag{8}$$

A Particular Beta Function Integral Representation[13]

$$\beta\left(\frac{x+1}{2},\frac{y+1}{2}\right) = 2\int_0^{\frac{\pi}{2}} \sin^x(\theta)\cos^y(\theta)\,d\theta \tag{9}$$

A Particular Beta Function Identity[14]

$$\beta(x,1-x) = \frac{\pi}{\sin(\pi x)} \tag{10}$$

Proof of Solution

$$I(a) = \int_0^\infty \frac{sin(x)}{x^a}\,dx \tag{11a}$$

$$\therefore\ f(x) = sin(x),\ g(x) = x^{-a} \tag{11b}$$

$$\therefore\ (8),(11b) \Longrightarrow \mathscr{L}\left(f(x)\right) = \frac{1}{x^2+1},\ \mathscr{L}^{-1}\left(g(x)\right) = \frac{x^{a-1}}{(a-1)!} \tag{11c}$$

$$\therefore\ (11c) \Longrightarrow \int_0^\infty \left(\frac{1}{x^2+1}\right)\left(\frac{x^{a-1}}{(a-1)!}\right)\,dx \tag{11d}$$

$$= \frac{1}{(a-1)!}\int_0^\infty \frac{x^{a-1}}{x^2+1}\,dx \tag{11e}$$

$$\therefore\ x = \tan(\theta) \Longrightarrow dx = sec^2(\theta)\,d\theta \tag{11f}$$

$$\therefore\ (11f) \Longrightarrow \int_0^{\frac{\pi}{2}} \tan^{a-1}(\theta)\,d\theta \tag{11g}$$

$$\therefore\ x = -y = a-1 \Longrightarrow \frac{\beta\left(\frac{a}{2},1-\frac{a}{2}\right)}{2(a-1)!} \tag{11h}$$

$$\therefore\ (10),(11h) \Longrightarrow \frac{\pi}{2(a-1)!}\csc\left(\frac{a\pi}{2}\right) \tag{11i}$$

$$= (7)\ \blacksquare \tag{11j}$$

Ep. 3 - General Quadratic Polynomial Divided By Another

$$\int \frac{ax^2+bx+K}{gx^2+hx+T}\,dx$$

Introduction

Polynomials of degree 2 (quadratics) have long been the turning point of Calculus I to Calculus II explorations. This integral aims to cover many aspects of Calculus II in one integral: trigonometric identities, logarithmic identities, and more.

$$I\left(a,b,K,g,h,T\right)=\int \frac{ax^2+bx+K}{gx^2+hx+T}\,dx \tag{12}$$

Proposed Solutions

You can find various integrals of this type online if you want to try/follow some more concrete examples. I also provided a different way to solve this by first utilizing *u-substitution*[15] before splitting the integral into 3 separate ones by their degree on my YouTube channel.[16] The first proof of the solution I have listed is done with *algebraic manipulation* as its priority. The second one, the far superior one in my opinion, utilizes *u-substitution* to vastly decrease the computation time. It allows for utilizing the identities with fewer steps. Simply compare the page space necessary to complete Method 1 - Algebraic Manipulation Priority (≈ 3 pg.'s) to Method 2 - U-Substitution Priority (≈ 1 pg.). As you'll see below the two primitives differ slightly, but actually yield the same result given a proper/definite integral.

$$\begin{aligned}I\left(a,b,K,g,h,T\right)&=a\left(x+\frac{h}{2g}\right)+\left(\frac{bg-ah}{2g}\right)\ln\left|gx^2+hx+T\right|\\&+\frac{K-\frac{bh+2aT}{2g}+2a\left(\frac{h}{2g}\right)^2}{\sqrt{\frac{T}{g}-\left(\frac{h}{2g}\right)^2}}\arctan\left(\frac{x+\frac{h}{2g}}{\sqrt{\frac{T}{g}-\left(\frac{h}{2g}\right)^2}}\right)+C\end{aligned} \tag{13}$$

Key Properties

General Arctangent Integral [17]

$$\int \frac{1}{(x+a)^2+b^2}\,dx = \frac{1}{b}\arctan\left(\frac{x+a}{b}\right)+C \tag{14}$$

General Natural Logarithmic Integral [18]

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)|+C \tag{15}$$

Proofs of Solutions

Method 1 - Algebraic Manipulation Priority

$$\text{Note}: \frac{d}{dx}\left(gx^2+hx+T\right) = 2gx+h \tag{16a}$$

$$\therefore\ I = \int \underbrace{\frac{ax^2}{gx^2+hx+T}}_{I_1} + \underbrace{\frac{bx}{gx^2+hx+T}}_{I_2} + \underbrace{\frac{K}{gx^2+hx+T}}_{I_3}\,dx \tag{16b}$$

$$\therefore\ (16b) \Longrightarrow I_3 = \int \frac{K}{gx^2+hx+T}\,dx \tag{16c}$$

$$\text{Note}: gx^2+hx+T = \frac{1}{g}\left(x^2+\frac{h}{g}x+\frac{T}{g}\right) = \frac{1}{g}\left(\left(x+\frac{h}{2g}\right)^2+\underbrace{\frac{T}{g}-\left(\frac{h}{2g}\right)^2}_{\Delta}\right) \tag{16d}$$

$$\therefore\ (16c),(16d) \Longrightarrow I_3 = \frac{K}{g}\int \frac{1}{\left(x+\frac{h}{2g}\right)^2+\Delta}\,dx \tag{16e}$$

$$\therefore\ (14),(16e) \Longrightarrow I_3 = \frac{K}{g\sqrt{\Delta}}\arctan\left(\frac{x+\frac{h}{2g}}{\sqrt{\Delta}}\right)+C_1 \tag{16f}$$

$$\therefore\ (16d),(16f) \Longrightarrow I_3 = \frac{K}{g\sqrt{\frac{T}{g}-\left(\frac{h}{2g}\right)^2}}\arctan\left(\frac{x+\frac{h}{2g}}{\sqrt{\frac{T}{g}-\left(\frac{h}{2g}\right)^2}}\right)+C_1 \tag{16g}$$

$$\therefore\ (16b) \Longrightarrow I_2 = \int \frac{bx}{gx^2+hx+T}\,dx = \frac{b}{2g}\int \frac{2gx+h-h}{gx^2+hx+T}\,dx \tag{16h}$$

$$= \underbrace{\frac{b}{2g}\int \frac{2gx+h}{gx^2+hx+T}\,dx}_{\text{use }(15)} - \underbrace{\frac{bh}{2g}\int \frac{1}{gx^2+hx+T}\,dx}_{\text{use }(16g)} \tag{16i}$$

$$\begin{aligned}
\therefore\ (15),(16g),(16i) \implies I_2 &= \left(\frac{b}{2g}\right)\left(\ln\left|gx^2+hx+T\right|+C_2\right)\\
&-\left(\frac{bh}{2g}\right)\left(\frac{1}{g\sqrt{\frac{T}{g}-\left(\frac{h}{2g}\right)^2}}\arctan\left(\frac{x+\frac{h}{2g}}{\sqrt{\frac{T}{g}-\left(\frac{h}{2g}\right)^2}}\right)+C_3\right) \qquad (16j)\\
&= \frac{b}{2g}\ln\left|gx^2+hx+T\right|+C_4\\
&-\frac{bh}{2g^2\sqrt{\frac{T}{g}-\left(\frac{h}{2g}\right)^2}}\arctan\left(\frac{x+\frac{h}{2g}}{\sqrt{\frac{T}{g}-\left(\frac{h}{2g}\right)^2}}\right)+C_5 \qquad (16k)\\
&= \frac{b}{2g}\ln\left|gx^2+hx+T\right|\\
&-\frac{bh}{2g^2\sqrt{\frac{T}{g}-\left(\frac{h}{2g}\right)^2}}\arctan\left(\frac{x+\frac{h}{2g}}{\sqrt{\frac{T}{g}-\left(\frac{h}{2g}\right)^2}}\right)+C_6 \qquad (16l)
\end{aligned}$$

$$\begin{aligned}
\therefore\ (16b) \implies I_1 &= \int \frac{ax^2}{gx^2+hx+T}\,dx \qquad (16m)\\
&= \frac{a}{g}\int \frac{gx^2+hx+T-(hx+T)}{gx^2+hx+T}\,dx \qquad (16n)\\
&= \frac{a}{g}\int \underbrace{\frac{gx^2+hx+T}{gx^2+hx+T}}_{=1}\,dx - \underbrace{\frac{a}{g}\int \frac{hx+T}{gx^2+hx+T}\,dx}_{I_2\text{ variant }+\,I_3\text{ variant}} \qquad (16o)
\end{aligned}$$

$$\therefore\ (16i),(16l),(16o) \implies I_1 = \frac{ax}{g} + C_7 - \left(\frac{a}{g}\right)\left(\frac{T}{g\sqrt{\frac{T}{g}-\left(\frac{h}{2g}\right)^2}}\arctan\left(\frac{x+\frac{h}{2g}}{\sqrt{\frac{T}{g}-\left(\frac{h}{2g}\right)^2}}\right)+C_8\right) - \left(\frac{a}{g}\right)\left(\frac{h}{2g}\ln\left|gx^2+hx+T\right|\right) + \left(\frac{a}{g}\right)\left(\frac{h^2}{2g^2\sqrt{\frac{T}{g}-\left(\frac{h}{2g}\right)^2}}\arctan\left(\frac{x+\frac{h}{2g}}{\sqrt{\frac{T}{g}-\left(\frac{h}{2g}\right)^2}}\right)+C_9\right) \tag{16p}$$

$$\therefore\ (16p) \implies I_1 = \frac{ax}{g} - \frac{ah}{2g^2}\ln\left|gx^2+hx+T\right| + \frac{\frac{ah^2}{2g^3}-\frac{aT}{g^2}}{\sqrt{\frac{T}{g}-\left(\frac{h}{2g}\right)^2}}\arctan\left(\frac{x+\frac{h}{2g}}{\sqrt{\frac{T}{g}-\left(\frac{h}{2g}\right)^2}}\right)+C_{10} \tag{16q}$$

$$\therefore\ (16b),(16g),(16l),(16q) \implies I = I_1 + I_2 + I_3 \tag{16r}$$

$$\therefore\ I = \frac{K}{g\sqrt{\frac{T}{g}-\left(\frac{h}{2g}\right)^2}}\arctan\left(\frac{x+\frac{h}{2g}}{\sqrt{\frac{T}{g}-\left(\frac{h}{2g}\right)^2}}\right)+C_1 + \frac{b}{2g}\ln\left|gx^2+hx+T\right| - \frac{bh}{2g^2\sqrt{\frac{T}{g}-\left(\frac{h}{2g}\right)^2}}\arctan\left(\frac{x+\frac{h}{2g}}{\sqrt{\frac{T}{g}-\left(\frac{h}{2g}\right)^2}}\right)+C_6 + \frac{ax}{g} - \frac{ah}{2g^2}\ln\left|gx^2+hx+T\right| + \frac{\frac{ah^2}{2g^3}-\frac{aT}{g^2}}{\sqrt{\frac{T}{g}-\left(\frac{h}{2g}\right)^2}}\arctan\left(\frac{x+\frac{h}{2g}}{\sqrt{\frac{T}{g}-\left(\frac{h}{2g}\right)^2}}\right)+C_{10} \tag{16s}$$

$$\therefore\ (16s) \implies I = \frac{ax}{g} + \left(\frac{b}{2g} - \frac{ah}{2g^2}\right)\ln\left|gx^2 + hx + T\right|$$
$$+ \frac{\frac{K}{g} - \frac{bh}{2g^2} + \frac{ah^2}{2g^3} - \frac{aT}{g^2}}{\sqrt{\frac{T}{g} - \left(\frac{h}{2g}\right)^2}} \arctan\left(\frac{x + \frac{h}{2g}}{\sqrt{\frac{T}{g} - \left(\frac{h}{2g}\right)^2}}\right) + C_{11} \tag{16t}$$
$$= (13)\ \blacksquare \tag{16u}$$

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Method 2 - U-Substitution Priority

$$\therefore\ u = x + \frac{h}{2g} \implies du = dx \tag{17a}$$

$$\therefore\ (12), (16d), (17a) \implies I = \int \frac{a\left(u - \frac{h}{2g}\right)^2 + b\left(u - \frac{h}{2g}\right) + K}{u^2 + \frac{T}{g} - \left(\frac{h}{2g}\right)^2}\,du \tag{17b}$$

$$= \int \frac{au^2 + u\left(b - \frac{ah}{g}\right) + K - \frac{bh}{2g} + a\left(\frac{h}{2g}\right)^2}{u^2 + \frac{T}{g} - \left(\frac{h}{2g}\right)^2}\,du \tag{17c}$$

$$= \int \overbrace{\frac{a\left(u^2 + \frac{T}{g} - \left(\frac{h}{2g}\right)^2\right)}{u^2 + \frac{T}{g} - \left(\frac{h}{2g}\right)^2}}^{a} + \overbrace{\frac{u\left(b - \frac{ah}{g}\right)}{u^2 + \frac{T}{g} - \left(\frac{h}{2g}\right)^2}}^{\therefore (15)}$$
$$+ \underbrace{\frac{K - \frac{bh}{2g} + a\left(\frac{h}{2g}\right)^2 - a\left(\frac{T}{g} - \left(\frac{h}{2g}\right)^2\right)}{u^2 + \frac{T}{g} - \left(\frac{h}{2g}\right)^2}}_{\therefore (14)}\,du \tag{17d}$$

$$\therefore\ (14), (15), (17d) \implies I = au + \left(\frac{bg - ah}{2g}\right)\ln\left|u^2 + \frac{T}{g} - \left(\frac{h}{2g}\right)^2\right|$$
$$+ \frac{K - \frac{bh + 2aT}{2g} + 2a\left(\frac{h}{2g}\right)^2}{\sqrt{\frac{T}{g} - \left(\frac{h}{2g}\right)^2}} \arctan\left(\frac{u}{\sqrt{\frac{T}{g} - \left(\frac{h}{2g}\right)^2}}\right) + C \tag{17e}$$

$$\therefore\ (17e) \Longrightarrow I = a\left(x + \frac{h}{2g}\right) + \left(\frac{bg - ah}{2g}\right)\ln\left|gx^2 + hx + T\right|$$

$$+ \frac{K - \frac{bh+2aT}{2g} + 2a\left(\frac{h}{2g}\right)^2}{\sqrt{\frac{T}{g} - \left(\frac{h}{2g}\right)^2}} \arctan\left(\frac{x + \frac{h}{2g}}{\sqrt{\frac{T}{g} - \left(\frac{h}{2g}\right)^2}}\right) + C \tag{17f}$$

$$= (13)\ \blacksquare \tag{17g}$$

Ep. 4 - Integration or Differentiation Under the Integral Sign?

$$I(a,b)=\int_0^1 \frac{x^b-x^a}{\ln(x)}\,dx$$

Introduction

As we've explored earlier, *differentiation under the integral sign* is a powerful tool in many scenarios, but have you ever employed the stark opposite utility: *integration under the integral sign*[19]? In all my time as a Mathematics student I have only used this in one class on my final exam and the reason for it was because our instructor forced us to use it as part of a proof. I, however, have recently been challenging myself to use this technique to solve certain tasking integrals. This particular integral is a famous example of how to use differentiation, but the integration version is far simpler on a conceptual basis.

$$I(a,b)=\int_0^1 \frac{x^b-x^a}{\ln(x)}\,dx \tag{18}$$

Proposed Solution

I will go about solving this by first using differerntiation then integration. As you'll soon see when using both you need to manipulate a base case to achieve the desired integral. I should note that there is an extension of the Leibniz Integral Rule to cover improper integrals. Go to the end of the PDF linked in end note 21's URL and Rob (the author) provides a small paragraph explaining the conditions necessary to use it.

$$I(a,b)=\ln\left|\frac{b+1}{a+1}\right| \tag{19}$$

Key Properties

A Useful Integral Identity [20]

$$\int_a^b dx \int_c^d f(x,\phi)\ d\phi = \int_c^d d\phi \int_a^b f(x,\phi)\ dx \tag{20}$$

Leibniz Integral Rule [21]

$$\frac{d}{dy}\int_a^b f(x,y)\ dx = \int_a^b \frac{\partial f}{\partial y}(x,y)\ dx \tag{21}$$

Proofs of Solution

Method 1 - Differentiation Under the Integral Sign

$$\text{Consider}: \int_0^1 \frac{x^a-1}{\ln(x)}\ dx \Longrightarrow I(0)=0 \tag{22a}$$

$$\therefore\ (21),(22a) \Longrightarrow I'(a) = \int_0^1 x^a\ dx = \frac{1}{a+1} \tag{22b}$$

$$\therefore\ (15),(22b) \Longrightarrow I(a) = \int \frac{1}{a+1}\ da = \ln|a+1| + C \tag{22c}$$

$$\therefore\ (22a),(22c) \Longrightarrow C=0 \Longrightarrow I(a) = \ln|a+1| \tag{22d}$$

$$\text{Note}: x^b - x^a = (x^b-1)-(x^a-1) \tag{22e}$$

$$\therefore\ (22c),(22e) \Longrightarrow \int_0^1 \frac{x^b-x^a}{\ln(x)}\ dx = \ln|b+1| - \ln|a+1| = \ln\left|\frac{b+1}{a+1}\right| \tag{22f}$$

$$= (19)\ \blacksquare \tag{22g}$$

Method 2 - Integration Under the Integral Sign

$$\text{Note}: \int_a^b x^\phi\ d\phi = \frac{x^b-x^a}{\ln|x|} \tag{23a}$$

$$\therefore\ (20),(23a) \Longrightarrow \int_0^1 \int_a^b x^\phi\ d\phi\ dx = \int_a^b \int_0^1 x^\phi\ dx\ d\phi \tag{23b}$$

$$\therefore\ (23a),(23b) \Longrightarrow \int_0^1 \frac{x^b-x^a}{\ln(x)}\ dx = \int_a^b \int_0^1 x^\phi\ dx\ d\phi \tag{23c}$$

$$\therefore\ (23c) \Longrightarrow \int_a^b \frac{1}{\phi+1}\ d\phi = \ln\left|\frac{b+1}{a+1}\right| \tag{23d}$$

$$= (19)\ \blacksquare \tag{23e}$$

Ep. 5 - Introducing Ramanujan's Master Theorem

$$I(a,b)=\int_0^1 x^a \ln^b(x)\,dx$$

Introduction

Ramanujan and Hardy created two mathematics identities that have largely disappeared from modern use: the *Hardy-Ramanujan Master Theorem*[22] and the *Ramanujan Interpolation Formula*[23]. With this integral I aim to show the intuitive networking of the Hardy-Ramanujan Master Theorem with many other mathematical identities like the Beta and Gamma Function.

$$I(a,b)=\int_0^1 x^a \ln^b(x)\,dx \tag{24}$$

Proposed Solution

I first encountered a similar integral on *Brilliant*[24] asking us to solve primarily if $a=b$ using Feynman's Technique and $a\neq b$ secondarily with whatever method possible. I only saw two solutions offered by those who had commented on the solutions wiki: Repeated Feynman and Gamma Function Manipulation. I saw an opportunity to use the Master Theorem and took it.

$$I(a,b)=\frac{(-1)^b(b)!}{(a+1)^{b+1}} \tag{25}$$

Key Properties

Hardy-Ramanujan Master Theorem[25]

$$\int_0^\infty x^{g-1} f(x)\, dx = \Gamma(g)\, \varphi(-g)\ ;\ f(x) = \sum_{n=0}^{\infty} \varphi(n) \frac{(-x)^n}{n!} \tag{26}$$

e^x Taylor Series[26]

$$e^x = \sum \frac{x^n}{n!} \tag{27}$$

Proof of Solution

$$\therefore\ x = e^{-u} \implies dx = -e^{-u}\, du \tag{28a}$$

$$\therefore\ (24), (28a) \implies (-1)^{-b} \int_0^\infty u^b\, e^{-u(a+1)}\, du \tag{28b}$$

$$\therefore\ (26), (27), (28b) \implies g = b + 1,\ \varphi(g) = (a+1)^g \tag{28c}$$

$$\therefore\ (28c) \implies I(a,b) = \frac{(-1)^b (b)!}{(a+1)^{b+1}} \tag{28d}$$

$$= (25)\ \blacksquare \tag{28e}$$

End Notes & Various Links

[1] *Introduction to the Gamma Function* by **Pascal Sebah** and **Xavier Gourdon**
See Ch. 8 The Beta Function on pg.17
`https://www.csie.ntu.edu.tw/~b89089/link/gammaFunction.pdf`

[2] *Ch.1 Hypergeometric Series* by **Andrew Sills**
`http://home.dimacs.rutgers.edu/~asills/teach/spr05/hypergeom.pdf`

[3] *Ch.3 Integral Transforms* by **Jose Farrill**
`https://www.maths.ed.ac.uk/~jmf/Teaching/MT3/IntegralTransforms.pdf`

[4] A primitive is a shorter way of saying antiderivatives in Mathematics.

[5] *Integrals: Royal Properties* by **C. D. Chester**
`https://cdchester.co.uk/2019/04/23/integrals-royal-properties/`

[6] Refer to 2

[7] blackpenredpen's YouTube Channel
`https://www.youtube.com/user/blackpenredpen`

[8] *Differentiation Under the Integral Sign* by **Leo Goldmakher**
`https://web.williams.edu/Mathematics/lg5/Feynman.pdf`

[9] *The main dish, integral of sin(x)/x from 0 to inf, via Feynman's Technique* by **blackpenredpen**
`https://www.youtube.com/watch?v=s1zhYD4x6mY`

[10] *A Treatment of the Dirichlet Integral via the Methods of Real Analysis* by **Guo Chen**
`https://www.math.uchicago.edu/~may/VIGRE/VIGRE2009/REUPapers/ChenGuo.pdf`

[11] *Laplace Transform*
See 4.4 Evaluating Integrals Over the Real Axis
`https://en.wikipedia.org/wiki/Laplace_transform`

[12] Refer to 11

[13] Refer to 1

[14] *Introduction to the Gamma Function* by **Pascal Sebah** and **Xavier Gourdon**
See Ch. 8.1 Special Values on pg.18
`https://www.csie.ntu.edu.tw/~b89089/link/gammaFunction.pdf`

[15] *Sec. 6.8 Integration by Substitution* by **Al Shenk**
See Theorem 1
`http://math.ucsd.edu/~ashenk/Section6_8.pdf`

[16] *Integrals Ep. 3 - General Quadratic Polynomial Divided By Another* by **C. D. Chester**
`https://www.youtube.com/watch?v=OaGB66-Zexw`

[17] *Sec. 6.8 Integration by Substitution* by **Al Shenk**
See pg. 256 for the adapted result
`http://math.ucsd.edu/~ashenk/Section6_8.pdf`

[18] *Natural Logarithm*
See Sec. 7 The Natural Logarithm in Integration
`https://en.wikipedia.org/wiki/Natural_logarithm`

[19] *Integration Under the Integral Sign*
`https://mathworld.wolfram.com/IntegrationUndertheIntegralSign.html`

[20] Refer to 19

[21] *MATH 203: The Leibniz Rule* by **Rob Harron**
`https://math.hawaii.edu/~rharron/teaching/MAT203/LeibnizRule.pdf`

[22] *A Note on the Ramanujan's Master Theorem* by **Lazhar Bougoffa**
See Theorem 3 on pg. 2
`https://arxiv.org/pdf/1902.01539.pdf`

[23] *Extension of Hardy's Class for Ramanujan's Interpolation Formula and Master Theorem with Applications* by **Muhammed Aslam Chaudhry** and **Asghar Qadir**
See (1.5) on pg. 2
`https://core.ac.uk/download/pdf/81057786.pdf`
[24] *Natural Logarithm Integral* by **Sameer Kailasa**
`https://brilliant.org/problems/natural-logarithm-integral/`
[25] Refer to 22
[26] *Commonly Used Taylor Series* by **Maria Girardi**
`http://people.math.sc.edu/girardi/m142/handouts/10sTaylorPolySeries.pdf`