Approximating Square Roots

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Outline

Acknowledgments & Prelude

I have seen various papers on computing the square root of a number, so I figured I would show you the four methods I would ever consider using in computing them. Those four respectively being: Integral Approximation, Householder's Method, Continued Fraction Expansion, and Intuitive Geometric Iteration. I must give credit to Matt Parker[1] (for showing Intuitive Geometric Iteration in a π day video[2] back in 2018), the user Rustyn[3] on Math Stack Exchange (for introducing me to Householder's Method[4] with an answer to a post on square root algorithms[5]), and Steve Chow[6] of YT as BPRP[7] (for showing me the general technique of the Continued Fraction Expansion in his video on Infinite Continued Fractions[8]). I would like to note that some of these (well technically all of these with some manipulation) can be expanded to be nth root finding algorithms.

Square Root Approximation Methods

Method 1 - Continued Fraction Expansion

$$n = a^2 + r \implies n - a^2 = r \implies (\sqrt{n} + a)(\sqrt{n} - a) = r \implies \sqrt{n} = a + \frac{r}{\sqrt{n} + a} \tag{1}$$

$$\therefore\ \sqrt{n} = a + \cfrac{r}{2a + \cfrac{r}{2a + \cfrac{r}{2a + \cfrac{r}{2a + \cfrac{r}{\cdots}}}}} \tag{2}$$

$$\tag{3}$$

This is probably the simplest one to use, in that it's just fractions and no advanced mathematics is necessary to derive it. Therefore, I'd probably use this method to first explain how we learned to approximate square roots and move on to things like Newton's Method after some other foundational mathematics had been explained. Here's an example of using this technique to estimate $\sqrt{2}$ and $\sqrt{5}$.

$$n = 2,\ a = 1,\ r = 1 \tag{4}$$

$$\therefore\ \sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{\cdots}}}}} \implies \frac{3}{2},\ \frac{7}{5},\ \frac{17}{12}, \cdots \tag{5}$$

$$n = 5,\ a = 2,\ r = 1 \tag{6}$$

$$\therefore\ \sqrt{5} = 2 + \cfrac{1}{4 + \cfrac{1}{4 + \cfrac{1}{4 + \cfrac{1}{4 + \cfrac{1}{\cdots}}}}} \implies \frac{9}{4},\ \frac{38}{17},\ \frac{161}{72}, \cdots \tag{7}$$

$$\tag{8}$$

Method 2 - Integral Approximation

$$\frac{d}{dx}\left(\sqrt{x}\right) = \frac{1}{2\sqrt{x}} \implies \int_0^a \frac{1}{2\sqrt{x}}\,dx = \sqrt{a} \tag{9}$$

This is probably the worst method on this list in my opinion. I say that because the *Open/Closed Newton-Cotes Formulas* that are used to approximate integrals use more square roots IE **circular reasoning**. However, it is still a normal method used in computers funnily enough. For these reasons, I will not be showing an example, rather just noting this *hole*.

Method 3 - Householder's Method

This is the family of approximations that Newton's and Halley's Method comes from. Newton's being $m = 1$ and Halley's being $m = 2$ respectively. Note that whatever the m-value used the iterations will converge at a rate $m + 1$. Therefore, Newton's is quadratic, Halley's is cubic, and so on.

$$\text{Let } g(x) = \frac{1}{f(x)} \tag{10}$$

$$x_{n+1} = x_n + m\left(\frac{g^{(m-1)}(x_n)}{g^{(m)}(x_n)}\right) \tag{11}$$

$$\therefore\ g^{(0)}(x) = \frac{1}{x^2 - a} \tag{12}$$

$$\therefore\ g^{(1)}(x) = \frac{-2x}{(x^2 - a)^2} \tag{13}$$

$$\therefore\ g^{(2)}(x) = \frac{2(3x^2 + a)}{(x^2 - a)^3} \tag{14}$$

$$\therefore\ g^{(3)}(x) = \frac{-24x(x^2 + a)}{(x^2 - a)^4} \tag{15}$$

$$\therefore\ g^{(4)}(x) = \frac{24(5x^4 + 10ax^2 + a^2)}{(x^2 - a)^5} \tag{16}$$

I would use $m = 4$ because it will save you many iterations if you want accuracy with a not so great approximation. This yields:

$$x_{n+1} = x_n - \frac{4x_n(x_n^4 - a^2)}{5x_n^4 + 10ax_n^2 + a^2} \tag{17}$$

$$\therefore\ \sqrt{a = 5} \implies x_0 = 2 \implies x_1 = 2 - \frac{4\cdot 2(2^4 - 5^2)}{5\cdot 2^4 + 10\cdot 5\cdot 2^2 + 5^2} = 2 + \frac{72}{305} = \frac{682}{305} \implies E \approx 3\times 10^{-6} \tag{18}$$

$$\therefore\ x_2 = \frac{2360712083917682}{1055742538989025} \implies E \approx 2\times 10^{-31} \tag{19}$$

$$\therefore\ x_3 = \frac{11731001709514722818863545287168345415565116127630814355308252224126311693426826}{5246263453328232337653161156373098592051018982775615738224368630675049121326 25} \implies E \approx 8\times 10^{-157} \tag{20}$$

Method 4 - Intuitive Geometric Iteration

I highly recommend you go an watch Matt Parker's video to better understand the iterative technique on a visual basis. will simply be showing how the iterations follow. I don't think Matt intended for it to be used this way, but it has been utilize as such. This is my personal favorite method to approximate square roots as it always over approximates.

$$\sqrt{a} = \sqrt{d^2 + c},\ x = \frac{c}{2d} \tag{21}$$

$$\varphi_1 = \frac{x^2}{2(d+x)} \Longrightarrow \varphi_2 = \frac{\varphi_1^2}{2(d+x-\varphi_1)} \Longrightarrow \varphi_3 = \frac{\varphi_2^2}{2(d+x-\varphi_1-\varphi_2)} \Longrightarrow \varphi_4 = \frac{\varphi_3^2}{2(d+x-\varphi_1-\varphi_2-\varphi_3)} \tag{22}$$

The correction terms φ_n follow this pattern above and decrease in size quite rapidly. Here's it approximating $\sqrt{5}$.

$$\sqrt{5} = \sqrt{2^2 + 1},\ \therefore\ x = \frac{1}{4} \tag{23}$$

$$\Longrightarrow \varphi_1 = \frac{x^2}{2(d+x)} = \frac{\frac{1}{16}}{2(2+\frac{1}{4})} = \frac{1}{16}\frac{2}{9} = \frac{1}{72} \tag{24}$$

$$\Longrightarrow \varphi_2 = \frac{\varphi_1^2}{2(d+x-\varphi_1)} = \frac{\frac{1}{5184}}{2(2+\frac{1}{4}-\frac{1}{72})} = \frac{1}{5184}\frac{36}{161} = \frac{1}{23184} \tag{25}$$

$$\Longrightarrow \varphi_3 = \frac{\varphi_2^2}{2(d+x-\varphi_1-\varphi_2)} = \frac{1}{2403763488} \tag{26}$$

$$\Longrightarrow \varphi_4 = \frac{\varphi_3^2}{2(d+x-\varphi_1-\varphi_2-\varphi_3)} = \frac{1}{25840354427429161536} \tag{27}$$

$$\therefore\ \sqrt{5} \approx 2 + \frac{1}{4} - \frac{1}{72} - \frac{1}{23184} - \frac{1}{2403763488} - \frac{1}{25840354427429161536} \Longrightarrow E \approx 4 \times 10^{-40} \tag{28}$$

References

[1]Matt Parker on Twitter
https://twitter.com/standupmaths?ref_src=twsrc%5Egoogle%7Ctwcamp%5Eserp%7Ctwgr%5Eauthor
[2]*How To Find A Square Root* by **standupmaths**
https://www.youtube.com/watch?v=Bwt5EZEb1Ns
[3]Rustyn's Math Stack Exchange Profile
https://math.stackexchange.com/users/53783/rustyn
[4]Householder's Method
https://en.wikipedia.org/wiki/Householder%27s_method
[5]Rustyn's Answer
https://math.stackexchange.com/a/296107
[6]See Steve Chow
http://www.piercecollege.edu/departments/mathematics/faculty.asp
[7]BPRP's Profile on YT
https://www.youtube.com/channel/UC_SvYP0k05UKiJ_2ndB02IA
[8]*Infinite Continued Fractions, Simple or Not?* by **BPRP**
https://www.youtube.com/watch?v=vw4eYXMWYEw