$$\int \frac{a\sin(x) + b\cos(x)}{m\sin(x) + n\cos(x)}\, dx$$

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Outline

Acknowledgments

Steve Chow, known more so online as BlackPenRedPen [1] (abbreviated as BPRP), created a video on YouTube (YT) titled *The 1, 2, 3, 4 Integral!* [2] in which he solves a trigonometric integral in a creative way that I had never seen before. I had known of these identities beforehand as they are basic Calculus I & II identities, but had never see them combined this way to start an integral. I'd seen them utilized after a bit of dissection onto an integral, but not this. After watching the video, back in early 2018, I scribbled down, in a notepad, how to generalize his integral to work for all values, not just 1 through 4 in that specific use. I can guarantee someone has already solved this integral, but I thought I'd show case the way Steve solves this integral and a couple other methods I thought of.

$$\int \frac{\sin(x) + 2\cos(x)}{3\sin(x) + 4\cos(x)}\, dx = \frac{1}{25}\left(11x + 2\ln|3\sin(x) + 4\cos(x)|\right) + C$$

Solutions

The following pages detail the methods I have found to solve this integral. The first method (and inspiration for this paper) uses simple identities to turn this integral into a solution using of a system of equations. It is by far the shortest and most intuitive method used. The second one uses a technique designed for trigonometric integrals: Weierstrass Substitution. It requires some Partial Fraction Decomposition to simplify, but once done it creates two simple integrals that can be solved with identities alone. The algebra is quite tedious however. The last method utilizes classic u-substitution and a clever use of 1 to transform the integral. Also a messy integral, but if you can learn to master identities it will go by quickly.

The BPRP Way

Steve uses a clever use of 1 and the principle integral identity of the natural logarithm simultaneously to solve this integral.[3]

$$\text{Note}: \underbrace{\int \frac{m\sin(x)+n\cos(x)}{m\sin(x)+n\cos(x)}\,dx}_{I_\alpha} = x + C \tag{1}$$

$$\text{Note}: \underbrace{\int \frac{-n\sin(x)+m\cos(x)}{m\sin(x)+n\cos(x)}\,dx}_{I_\beta} = \ln|m\sin(x)+n\cos(x)| + C \tag{2}$$

$$\therefore\ \kappa(I_\alpha)+\lambda(I_\beta) = \int \frac{a\sin(x)+b\cos(x)}{m\sin(x)+n\cos(x)}\,dx \tag{3}$$

$$\therefore\ \underbrace{\kappa m - \lambda n = a}_{\phi}\,,\ \underbrace{\kappa n + \lambda m = b}_{\varphi} \tag{4}$$

$$\therefore\ m\phi + n\varphi \implies \kappa m^2 + \kappa n^2 = am + bn \implies \kappa = \frac{am+bn}{m^2+n^2} \tag{5}$$

$$\therefore\ (4),(5) \implies \left(\frac{am+bn}{m^2+n^2}\right)m - \lambda n = a \implies -\lambda n = a - \left(\frac{am+bn}{m^2+n^2}\right)m \tag{6}$$

$$\therefore\ (6) \implies \lambda = \left(\frac{am+bn}{m^2+n^2}\right)\left(\frac{m}{n}\right) - \frac{a}{n} \tag{7}$$

$$\therefore\ (7) \implies \left(\frac{am+bn}{m^2+n^2}\right)x + \left[\left(\frac{am+bn}{m^2+n^2}\right)\left(\frac{m}{n}\right) - \frac{a}{n}\right]\ln|m\sin(x)+n\cos(x)| + C \tag{8}$$

Weierstrass Substitution

Weierstrass Substitution (WS) [4] is usually a viable technique to apply (if you don't mind the algebra) when dealing with trigonometric integrals.

$$I = \int \frac{a\sin(x) + b\cos(x)}{m\sin(x) + n\cos(x)}\, dx \tag{9}$$

$$\therefore\ u = \tan\left(\frac{x}{2}\right) \implies dx = \left(\frac{2}{1+u^2}\right)\, du \tag{10}$$

$$\therefore\ \int \frac{a\left(\frac{2}{1+u^2}\right) + b\left(\frac{1-u^2}{1+u^2}\right)}{m\left(\frac{2}{1+u^2}\right) + n\left(\frac{1-u^2}{1+u^2}\right)} \left(\frac{2}{1+u^2}\right)\, du \tag{11}$$

$$= 2\int \frac{a\left(\frac{2}{1+u^2}\right) + b\left(\frac{1-u^2}{1+u^2}\right)}{2m + n(1-u^2)}\, du \tag{12}$$

$$= 2\int \frac{2a + b(1-u^2)}{(1+u^2)(2m + n(1-u^2))}\, du \tag{13}$$

Now performing Partial Fraction Decomposition (PFD):

$$\frac{2a + b(1-u^2)}{\cdots} \implies \frac{2a + b - bu^2}{\cdots} = \frac{Au + B}{1+u^2} + \frac{Mu + N}{2m + n(1-u^2)} \tag{14}$$

$$\therefore\ 2a + b - bu^2 =$$
$$u^3(-An + M) + u^2(-Bn + N) + u(2Am + An + M) + (2Bm + Bn + N) \tag{15}$$

$$\varphi_1 \implies -An + M = 0\ ,$$
$$\varphi_2 \implies -Bn + N = -b\ ,$$
$$\varphi_3 \implies 2Am + An + M = 0\ ,$$
$$\varphi_4 \implies 2Bm + Bn + N = 2a + b \tag{16}$$

$$\therefore\ \varphi_1 \implies M = An \therefore\ \varphi_3 \implies 2Am + An + An = 0 \implies A = 0 \implies M = 0 \tag{17}$$

$$\therefore\ \varphi_4 - \varphi_2 \implies 2B(m+n) = 2(a+b) \implies B = \frac{a+b}{m+n} \tag{18}$$

$$\therefore\ \varphi_4 \implies N = 2a + b - \left(\frac{a+b}{m+n}\right)(2m+n) \tag{19}$$

$$\therefore\ (17), (18), (19) \implies A = 0,\ B = \frac{a+b}{m+n},\ M = 0,\ N = 2a + b - \left(\frac{a+b}{m+n}\right)(2m+n) \tag{20}$$

Now, using the four newly found values turns that original quartic from our WS into two simple quadratic integrals.

$$\therefore\ \frac{I}{2} = \underbrace{\int \frac{\left(\frac{a+b}{m+n}\right)}{1+u^2}\,du}_{I_1} + \underbrace{\int \frac{\left(2a+b-\left(\frac{a+b}{m+n}\right)(2m+n)\right)}{2m+n-nu^2}\,du}_{I_2} \tag{21}$$

$$\therefore\ I_1 = \left(\frac{a+b}{m+n}\right)\arctan(u) = \left(\frac{a+b}{m+n}\right)\arctan(\tan\left(\frac{x}{2}\right)) = \left(\frac{a+b}{m+n}\right)\left(\frac{x}{2}\right) + C \tag{22}$$

$$\therefore\ I_2 = \left(2a+b-\left(\frac{a+b}{m+n}\right)(2m+n)\right)\int \frac{1}{2m+n-nu^2}\,du \tag{23}$$

$$\therefore\ g = \sqrt{n}u \implies du = \frac{dg}{\sqrt{n}} \tag{24}$$

$$\therefore\ I_2 \implies \left(2a+b-\left(\frac{a+b}{m+n}\right)(2m+n)\right)\left(\frac{1}{\sqrt{n}}\right)\int \frac{1}{2m+n-g^2}\,dg \tag{25}$$

$$= \left(2a+b-\left(\frac{a+b}{m+n}\right)(2m+n)\right)\left(\frac{1}{\sqrt{n(2m+n)}}\right)\tanh^{-1}\left(\frac{g}{\sqrt{2m+n}}\right) \tag{26}$$

$$= \left(2a+b-\left(\frac{a+b}{m+n}\right)(2m+n)\right)\left(\frac{1}{\sqrt{n(2m+n)}}\right)\tanh^{-1}\left(u\sqrt{\frac{n}{2m+n}}\right) \tag{27}$$

$$= \left(2a+b-\left(\frac{a+b}{m+n}\right)(2m+n)\right)\left(\frac{1}{\sqrt{n(2m+n)}}\right)\tanh^{-1}\left(\tan\left(\frac{x}{2}\right)\sqrt{\frac{n}{2m+n}}\right) + C \tag{28}$$

$$\therefore\ I = 2(I_1 + I_2) \tag{29}$$

$$= \left(\frac{a+b}{m+n}\right)x + \left(2a+b-\left(\frac{a+b}{m+n}\right)(2m+n)\right)\left(\frac{2}{\sqrt{n(2m+n)}}\right)\tanh^{-1}\left(\tan\left(\frac{x}{2}\right)\sqrt{\frac{n}{2m+n}}\right) + C \tag{30}$$

Classic Substitution

$$I = \int \frac{a\sin(x)+b\cos(x)}{m\sin(x)+n\cos(x)}\,dx = \int \frac{(a\sin(x)+b\cos(x))(m\sin(x)-n\cos(x))}{m^2\sin^2(x)-n^2\cos^2(x)}\,dx \tag{31}$$

$$= \int \frac{am\sin^2(x)-bn\cos^2(x)-an\sin(x)\cos(x)+bm\sin(x)\cos(x)}{m^2\sin^2(x)-n^2\cos^2(x)}\,dx \tag{32}$$

$$= \int \frac{am(1-\cos^2(x))-bn\cos^2(x)+(bm-an)\sin(x)\cos(x)}{m^2(1-\cos^2(x))-n^2\cos^2(x)}\,dx \tag{33}$$

$$= \int \frac{am-(am+bn)\cos^2(x)+(bm-an)\sin(x)\cos(x)}{m^2-(m^2+n^2)\cos^2(x)}\,dx \tag{34}$$

$$= \int \frac{am-(am+bn)(\frac{\cos(2x)+1}{2})+(\frac{bm-an}{2})\sin(2x)}{m^2-(m^2+n^2)(\frac{\cos(2x)+1}{2})}\,dx \tag{35}$$

$$= \int \frac{am-(\frac{am+bn}{2})-(\frac{am+bn}{2})\cos(2x)+(\frac{bm-an}{2})\sin(2x)}{m^2-(\frac{m^2+n^2}{2})-(\frac{m^2+n^2}{2})\cos(2x)}\,dx \tag{36}$$

$$\therefore\ u = 2x \implies dx = \frac{du}{2} \tag{37}$$

$$\therefore\ \left(\frac{1}{2}\right)\int \frac{am-(\frac{am+bn}{2})-(\frac{am+bn}{2})\cos(u)+(\frac{bm-an}{2})\sin(u)}{m^2-(\frac{m^2+n^2}{2})-(\frac{m^2+n^2}{2})\cos(u)}\,du \tag{38}$$

$$= \left(\frac{1}{2}\right)\int \frac{2am-(am+bn)-(am+bn)\cos(u)+(bm-an)\sin(u)}{2m^2-(m^2+n^2)-(m^2+n^2)\cos(u)}\,du \tag{39}$$

$$I = \underbrace{\int \frac{am-(\frac{am+bn}{2})}{2m^2-(m^2+n^2)-(m^2+n^2)\cos(u)}\,du}_{I_1} + \underbrace{\int \frac{-(\frac{am+bn}{2})\cos(u)}{2m^2-(m^2+n^2)-(m^2+n^2)\cos(u)}\,du}_{I_2}$$

$$+ \underbrace{\left(\frac{bm-an}{2}\right)\int \frac{\sin(u)}{2m^2-(m^2+n^2)-(m^2+n^2)\cos(u)}\,du}_{I_3} \tag{40}$$

$$\therefore\ I_3 = \left(\frac{bm-an}{2(m^2+n^2)}\right)\ln\left|2m^2-(m^2+n^2)-(m^2+n^2)\cos(2x)\right| + C \tag{41}$$

$$\therefore\ I_2 = \int \frac{-(\frac{am+bn}{2})\cos(u)}{2m^2-(m^2+n^2)-(m^2+n^2)\cos(u)}\ du \tag{42}$$

$$= \left(\frac{am+bn}{2(m^2+n^2)}\right)\int \frac{2m^2-(m^2+n^2)-(m^2+n^2)\cos(u)-(2m^2-(m^2+n^2))}{2m^2-(m^2+n^2)-(m^2+n^2)\cos(u)}\ du \tag{43}$$

$$= \left(\frac{am+bn}{2(m^2+n^2)}\right)x - \underbrace{\int \frac{\frac{(am+bn)(2m^2-(m^2+n^2))}{2(m^2+n^2)}}{2m^2-(m^2+n^2)-(m^2+n^2)\cos(u)}\ du}_{I_4} \tag{44}$$

$$\therefore\ I_1 - I_4 = \left(am - \frac{am+bn}{2} - \frac{(am+bn)(2m^2-(m^2+n^2))}{2(m^2+n^2)}\right)\underbrace{\int(\cdots)\ du}_{I_5} \tag{45}$$

$$\therefore\ I_5 = \int \frac{1}{2m^2-(m^2+n^2)-(m^2+n^2)\cos(u)}\ du \tag{46}$$

$$\therefore\ g = \tan\left(\frac{u}{2}\right) \Longrightarrow du = \left(\frac{2}{1+g^2}\right)\ dg \tag{47}$$

$$\therefore\ (46),(47) \Longrightarrow \int \frac{1}{2m^2-(m^2+n^2)-(m^2+n^2)\left(\frac{1-g^2}{1+g^2}\right)}\left(\frac{2}{1+g^2}\right)\ dg \tag{48}$$

$$= 2\int \frac{1}{(2m^2-(m^2+n^2))(1+g^2)-(m^2+n^2)(1-g^2)}\ dg \tag{49}$$

$$= \int \frac{1}{(gm)^2-n^2}\ dg \tag{50}$$

$$= -\frac{\tanh^{-1}(\frac{gm}{n})}{mn} = -\frac{\tanh^{-1}(\frac{m}{n}\tan(x))}{mn} \tag{51}$$

$$\therefore\ I = I_1 + I_2 + I_3 \tag{52}$$

$$= \left(\frac{am+bn}{2(m^2+n^2)}\right)x + \left(\frac{bm-an}{2(m^2+n^2)}\right)\ln\left|2m^2-(m^2+n^2)-(m^2+n^2)\cos(2x)\right.$$
$$-\left(am - \frac{am+bn}{2} - \frac{(am+bn)(2m^2-(m^2+n^2))}{2(m^2+n^2)}\right)\left(\frac{1}{mn}\right)\tanh^{-1}\left(\frac{m}{n}\tan(x)\right) + C \tag{53}$$

References

[1]BPRP's YouTube Channel
`https://www.youtube.com/user/blackpenredpen`
[2]*The 1, 2, 3, 4 Integral!* by BPRP
`https://youtu.be/H2PtO5ayHl8`
[3]*Some "Tricks" for Integration* by **Steve Cohn**
`https://www.math.unl.edu/~scohn1/EngRevf08/integrationtricks.pdf`
[4]*Math 113: The Weierstrass Substitution* by **Dr. Scott Hyde**
`https://jekyll.math.byuh.edu/courses/m113/handouts/weierstrass.pdf`