

## Area Under $y = x^n$ From 0 to 1: Geometry vs. Integral by C. D. Chester

### PART 1: GEOMETRIC ADDITION PROBLEM

I have been watching a lot of BlackPenRedPen (BPRP for short) videos as of late. My main interest has been finding a formula for:

$$S_k = \sum_{n=1}^k (n)^k; n, k \in \mathbb{N}$$

In my “quest” I stumbled upon one of his guest videos<sup>1</sup> where his guest Max used geometry and a difference equation to show the solutions to  $k = 2, 3$  in the above formula.<sup>2</sup>

The image shows a whiteboard with handwritten mathematical work. At the top, the binomial expansion is written:  $(n+1)^4 - n^4 = 4n^3 + 6n^2 + 4n + 1$ . Below this, several instances of the equation are shown for  $n=1, 2, 3, \dots, m$ , with the terms  $2^4, 3^4, 4^4, \dots, m^4$  crossed out. The final equation is  $(m+1)^4 - 1 = 4(1^3 + 2^3 + \dots + m^3) + 6S_2 + 4S_1 + S_0$ , where the sum of cubes is boxed and labeled  $S_3$ .

$$(n+1)^4 - n^4 = 4n^3 + 6n^2 + 4n + 1$$
$$\cancel{2^4} - 1^4 = 4 \cdot 1^3 + 6 \cdot 1^2 + 4 \cdot 1 + 1$$
$$3^4 - \cancel{2^4} = 4 \cdot 2^3 + 6 \cdot 2^2 + 4 \cdot 2 + 1$$
$$\cancel{4^4} - \cancel{3^4} = 4 \cdot 3^3 + 6 \cdot 3^2 + 4 \cdot 3 + 1$$
$$\vdots$$
$$(m+1)^4 - \cancel{m^4} = 4 \cdot m^3 + 6 \cdot m^2 + 4 \cdot m + 1$$
$$(m+1)^4 - 1 = 4 \left( \underbrace{1^3 + 2^3 + \dots + m^3}_{S_3} \right) + 6 \cdot S_2 + 4 \cdot S_1 + S_0$$

<sup>1</sup> “ $1^3 + 2^3 + 3^3 + \dots + n^3$  and its geometry”

<https://www.youtube.com/watch?v=pxYhN3hvKXM&t=368s>

<sup>2</sup> I just wanted to note that Max prefers the geometry way.

In a different video he shows the solution to  $k = 1$ :

$$S_1 = \sum_{n=1}^n n = \frac{n(n+1)}{2}$$

Here is the technique I gathered from Max's explanations. I will start with  $k = 1$  to show the quasi-recurrence formula I have found.

$$(n+1)^2 = n^2 + 2n + 1$$

$$(n+1)^2 - n^2 = 2n + 1$$

Using the technique shown above in the picture yields:

$$(n+1)^2 - 1 = 2 \sum_{n=1}^n n + \sum_{n=1}^n 1$$

$$(n+1)^2 - 1 = 2S_1 + S_0$$

$$(n+1)^2 - 1 = 2S_1 + n$$

$$2S_1 = (n+1)^2 - n - 1$$

$$2S_1 = n^2 + 2n + 1 - n - 1$$

$$2S_1 = n^2 + n = n(n+1)$$

$$S_1 = \frac{n(n+1)}{2}$$

Now for  $k = 2$ :

$$(n+1)^3 = n^3 + 3n^2 + 3n + 1$$

$$(n+1)^3 - n^3 = 3n^2 + 3n + 1$$

$$\therefore (n+1)^3 - 1 = 3 \sum_{n=1}^n n^2 + 3 \sum_{n=1}^n n + \sum_{n=1}^n 1$$

$$(n+1)^3 - 1 = 3S_2 + 3S_1 + S_0$$

With some algebra yields:

$$S_2 = \frac{n(n+1)(2n+1)}{6}$$

In the general case you will see a pattern appear. The pattern looks like this:

$$(n+1)^{k+1} - 1 = \binom{k+2}{2} \sum_{n=1}^n n^k + \binom{k+1}{3} \sum_{n=1}^n n^{k-1} + \dots + \binom{k+2}{k+2} \sum_{n=1}^n n^0$$

$$(n+1)^{k+1} - 1 = \binom{k+2}{2} S_k + \binom{k+1}{3} S_{k-1} + \dots + \binom{k+2}{k+2} S_0$$

$$(n+1)^{k+1} - 1 = \binom{k+2}{2} S_k + \sum_{g=1}^k \binom{k+2}{g+2} S_{k-g}$$

Note this formula assumes  $k \geq 2$  and with some manipulation:

$$S_k = \frac{(n+1)^{k+1} - 1 - \sum_{g=1}^k \binom{k+2}{g+2} S_{k-g}}{\binom{k+2}{2}}$$

## PART 2: AREA UNDER $y = x^n$ FROM 0 TO 1

**$n = 1$  via Rectangular Sum**

$$Area = \lim_{n \rightarrow \infty} \left[ \frac{1}{n} \left( \left(\frac{1}{n}\right)^1 + \left(\frac{2}{n}\right)^1 + \dots + \left(\frac{n}{n}\right)^1 \right) \right] = \lim_{n \rightarrow \infty} \left( \frac{S_1}{n^2} \right) = \frac{1}{2}$$

**$n = 2$  via Rectangular Sum**

$$Area = \lim_{n \rightarrow \infty} \left[ \frac{1}{n} \left( \left(\frac{1}{n}\right)^2 + \left(\frac{2}{n}\right)^2 + \dots + \left(\frac{n}{n}\right)^2 \right) \right] = \lim_{n \rightarrow \infty} \left( \frac{S_2}{n^3} \right) = \frac{1}{3}$$

**$n = k$  via Rectangular Sum**

$$Area = \lim_{n \rightarrow \infty} \left[ \frac{1}{n} \left( \left(\frac{1}{n}\right)^k + \left(\frac{2}{n}\right)^k + \dots + \left(\frac{n}{n}\right)^k \right) \right] = \lim_{n \rightarrow \infty} \left( \frac{S_k}{n^{k+1}} \right)$$

**$n = k$  via Integral**

$$\int_0^1 x^k dx = \frac{x^{k+1}}{k+1} \Big|_0^1 = \frac{1}{k+1}$$

$$\therefore \lim_{n \rightarrow \infty} \left( \frac{S_k}{n^{k+1}} \right) = \frac{1}{k+1}$$