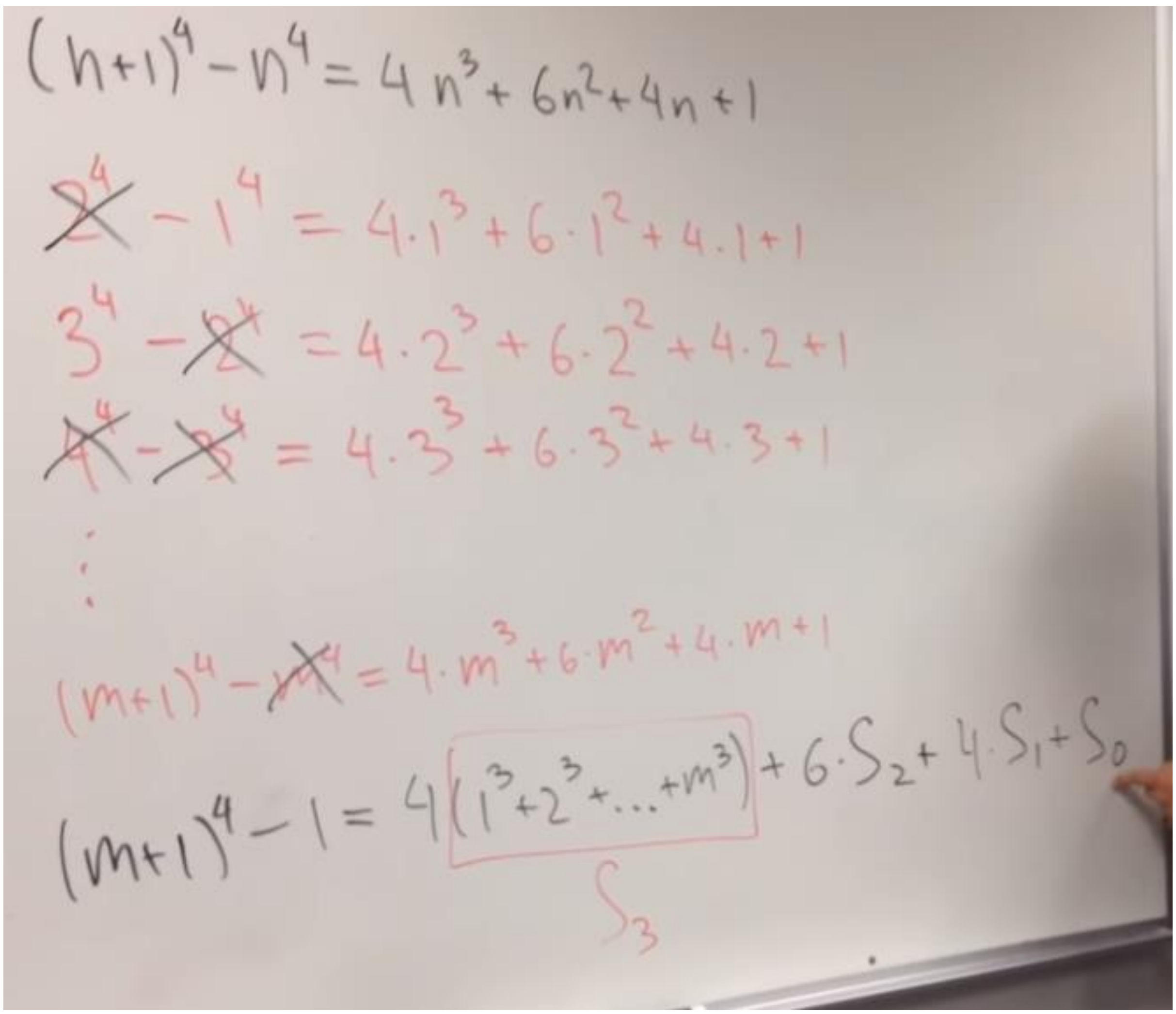
Area Under $y = x^n$ From 0 to 1: Geometry vs. Integral **by C. D. Chester**

PART 1: GEOMETRIC ADDITION PROBLEM

I have been watching a lot of BlackPenRedPen (BPRP for short) videos as of late. My main interest has been finding a formula for:

$$S_k = \sum_{n=1}^{n} (n)^k \, ; n, k \in \mathbb{N}$$

In my "quest" I stumbled upon one of his guest videos[1] where his guest Max used geometry and a difference equation to show the solutions to $k = 2{,}3$ in the above formula.[2]



---

[1] "1^3+2^3+3^3+...+n^3 and its geometry" https://www.youtube.com/watch?v=pxYhN3hvKXM&t=368s

[2] I just wanted to note that Max prefers the geometry way.

In a different video he shows the solution to $k = 1$:

$$S_1 = \sum_{n=1}^{n} n = \frac{n(n+1)}{2}$$

Here is the technique I gathered from Max's explanations. I will start with $k = 1$ to show the quazi-recurrence formula I have found.

$$(n+1)^2 = n^2 + 2n + 1$$
$$(n+1)^2 - n^2 = 2n + 1$$

Using the technique shown above in the picture yields:

$$(n+1)^2 - 1 = 2\sum_{n=1}^{n} n + \sum_{n=1}^{n} 1$$
$$(n+1)^2 - 1 = 2S_1 + S_0$$
$$(n+1)^2 - 1 = 2S_1 + n$$
$$2S_1 = (n+1)^2 - n - 1$$
$$2S_1 = n^2 + 2n + 1 - n - 1$$
$$2S_1 = n^2 + n = n(n+1)$$
$$S_1 = \frac{n(n+1)}{2}$$

Now for $k = 2$:

$$(n+1)^3 = n^3 + 3n^2 + 3n + 1$$
$$(n+1)^3 - n^3 = 3n^2 + 3n + 1$$
$$\therefore (n+1)^3 - 1 = 3\sum_{n=1}^{n} n^2 + 3\sum_{n=1}^{n} n + \sum_{n=1}^{n} 1$$
$$(n+1)^3 - 1 = 3S_2 + 3S_1 + S_0$$

With some algebra yields:

$$S_2 = \frac{n(n+1)(2n+1)}{6}$$

In the general case you will see a pattern appear. The pattern looks like this:

$$(n+1)^{k+1} - 1 = \binom{k+2}{2}\sum_{n=1}^{n} n^k + \binom{k+1}{3}\sum_{n=1}^{n} n^{k-1} + \cdots + \binom{k+2}{k+2}\sum_{n=1}^{n} n^0$$
$$(n+1)^{k+1} - 1 = \binom{k+2}{2} S_k + \binom{k+1}{3} S_{k-1} + \cdots + \binom{k+2}{k+2} S_0$$
$$(n+1)^{k+1} - 1 = \binom{k+2}{2} S_k + \sum_{g=1}^{k} \binom{k+2}{g+2} S_{k-g}$$

Note this formula assumes $k \geq 2$ and with some manipulation:

$$S_k = \frac{(n+1)^{k+1} - 1 - \sum_{g=1}^{k} \binom{k+2}{g+2} S_{k-g}}{\binom{k+2}{2}}$$

PART 2: AREA UNDER $y = x^n$ FROM 0 TO 1

$n = 1$ via Rectangular Sum

$$Area = \lim_{n\to\infty} \left[\frac{1}{n}\left(\left(\frac{1}{n}\right)^1 + \left(\frac{2}{n}\right)^1 + \cdots + \left(\frac{n}{n}\right)^1\right)\right] = \lim_{n\to\infty} \left(\frac{S_1}{n^2}\right) = \frac{1}{2}$$

$n = 2$ via Rectangular Sum

$$Area = \lim_{n\to\infty} \left[\frac{1}{n}\left(\left(\frac{1}{n}\right)^2 + \left(\frac{2}{n}\right)^2 + \cdots + \left(\frac{n}{n}\right)^2\right)\right] = \lim_{n\to\infty} \left(\frac{S_2}{n^3}\right) = \frac{1}{3}$$

$n = k$ via Rectangular Sum

$$Area = \lim_{n\to\infty} \left[\frac{1}{n}\left(\left(\frac{1}{n}\right)^k + \left(\frac{2}{n}\right)^k + \cdots + \left(\frac{n}{n}\right)^k\right)\right] = \lim_{n\to\infty} \left(\frac{S_k}{n^{k+1}}\right)$$

$n = k$ via Integral

$$\int_0^1 x^k dx = \frac{x^{k+1}}{k+1}\Bigg|_0^1 = \frac{1}{k+1}$$

$$\therefore \lim_{n\to\infty} \left(\frac{S_k}{n^{k+1}}\right) = \frac{1}{k+1}$$