Approximating Square Roots Using Geometric Iterations

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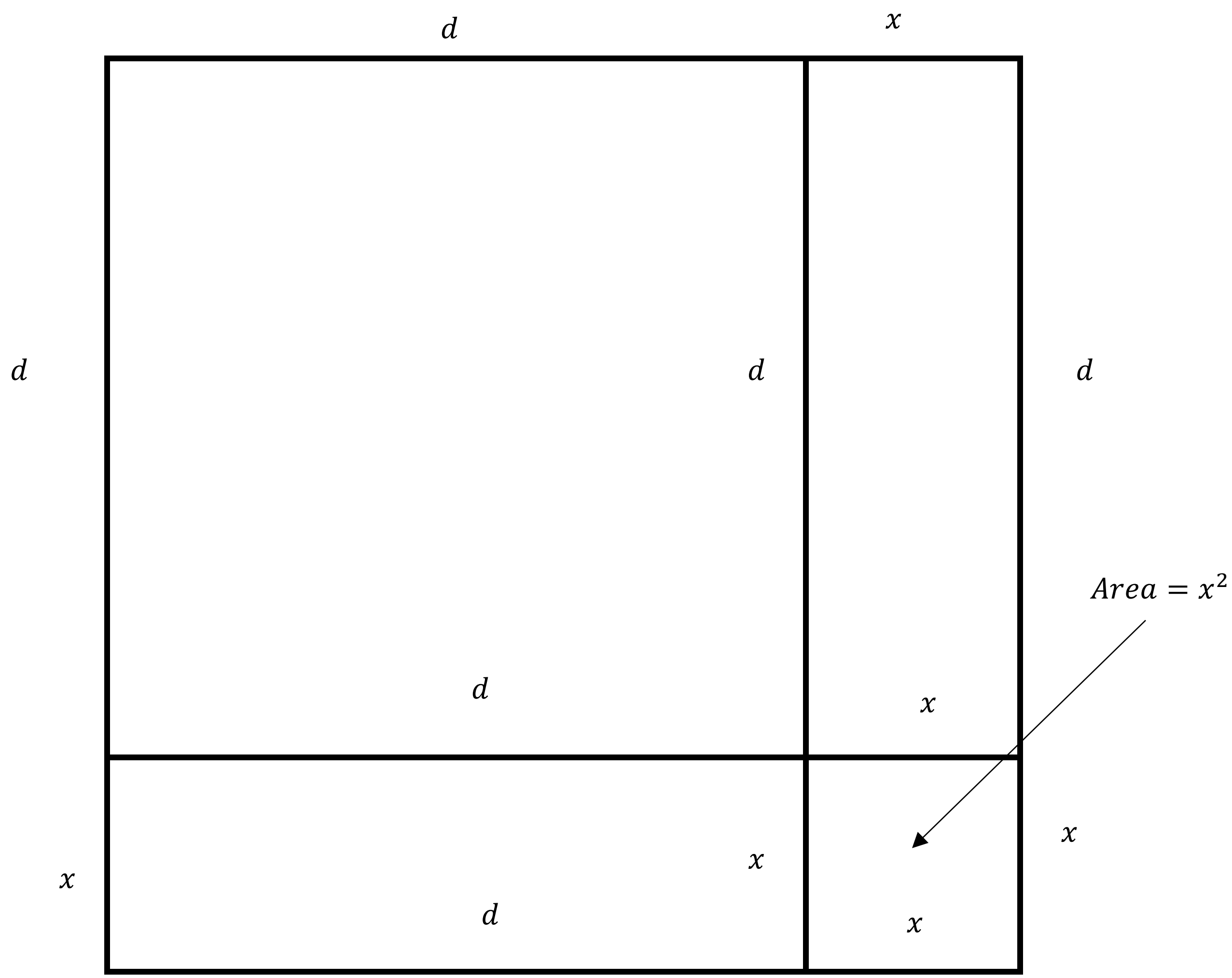
Imagine you have a number a and you want to take its square root.

Thus, $\sqrt{a} = \sqrt{b+c}$ where $b = \sqrt{d^2}$.

Therefore, $\sqrt{a} = \sqrt{\sqrt{d^2} + c}$.

A geometric approximation of this is found below:



As you can see there is an error of x^2. Note $dx = \frac{c}{2} \to x = \frac{c}{2d}$.

From there you subtract via two rectangles an area of x^2 using an error variable noted as y.

$$\therefore (d + x)y = \frac{x^2}{2} \to y = \frac{x^2}{2(d + x)} = \frac{\left(\frac{c}{2d}\right)^2}{2\left(d + \frac{c}{2d}\right)}$$

Repeating for another error reduction using another error variable z:

$$\therefore (d + x - y)z = \frac{y^2}{2} \to z = \frac{y^2}{2(d + x - y)} = \frac{\left(\frac{\left(\frac{c}{2d}\right)^2}{2\left(d + \frac{c}{2d}\right)}\right)^2}{2\left(d + \left(\frac{c}{2d}\right) - \left(\frac{\left(\frac{c}{2d}\right)^2}{2\left(d + \frac{c}{2d}\right)}\right)\right)}$$

Repeating for another error reduction using another error variable δ:

$$\therefore (d+x-y-z)\delta = \frac{z^2}{2} \rightarrow \delta = \frac{z^2}{2(d+x-y-z)} = \frac{\left(\frac{\left(\frac{\left(\frac{c}{2d}\right)^2}{2\left(d+\frac{c}{2d}\right)}\right)^2}{2\left(d+\left(\frac{c}{2d}\right)-\left(\frac{\left(\frac{c}{2d}\right)^2}{2\left(d+\frac{c}{2d}\right)}\right)\right)}\right)^2}{2\left(d+\left(\frac{c}{2d}\right)-\left(\frac{\left(\frac{c}{2d}\right)^2}{2\left(d+\frac{c}{2d}\right)}\right)-\left(\frac{\left(\frac{\left(\frac{c}{2d}\right)^2}{2\left(d+\frac{c}{2d}\right)}\right)^2}{2\left(d+\left(\frac{c}{2d}\right)-\left(\frac{\left(\frac{c}{2d}\right)^2}{2\left(d+\frac{c}{2d}\right)}\right)\right)}\right)\right)}$$

Substituting this back in to approximate $\sqrt{a}$ yields:

$$\sqrt{a} = d + x - y - z - \delta$$

$$\sqrt{a} = d + \left(\frac{c}{2d}\right) - \left(\frac{\left(\frac{c}{2d}\right)^2}{2\left(d+\left(\frac{c}{2d}\right)\right)}\right) - \left(\frac{\left(\frac{\left(\frac{c}{2d}\right)^2}{2\left(d+\left(\frac{c}{2d}\right)\right)}\right)^2}{2\left(d+\left(\frac{c}{2d}\right)-\left(\frac{\left(\frac{c}{2d}\right)^2}{2\left(d+\left(\frac{c}{2d}\right)\right)}\right)\right)}\right) - \left(\frac{\left(\frac{\left(\frac{\left(\frac{c}{2d}\right)^2}{2\left(d+\left(\frac{c}{2d}\right)\right)}\right)^2}{2\left(d+\left(\frac{c}{2d}\right)-\left(\frac{\left(\frac{c}{2d}\right)^2}{2\left(d+\left(\frac{c}{2d}\right)\right)}\right)\right)}\right)^2}{2\left(d+\left(\frac{c}{2d}\right)-\left(\frac{\left(\frac{c}{2d}\right)^2}{2\left(d+\left(\frac{c}{2d}\right)\right)}\right)-\left(\frac{\left(\frac{\left(\frac{c}{2d}\right)^2}{2\left(d+\left(\frac{c}{2d}\right)\right)}\right)^2}{2\left(d+\left(\frac{c}{2d}\right)-\left(\frac{\left(\frac{c}{2d}\right)^2}{2\left(d+\left(\frac{c}{2d}\right)\right)}\right)\right)}\right)\right)}\right)$$