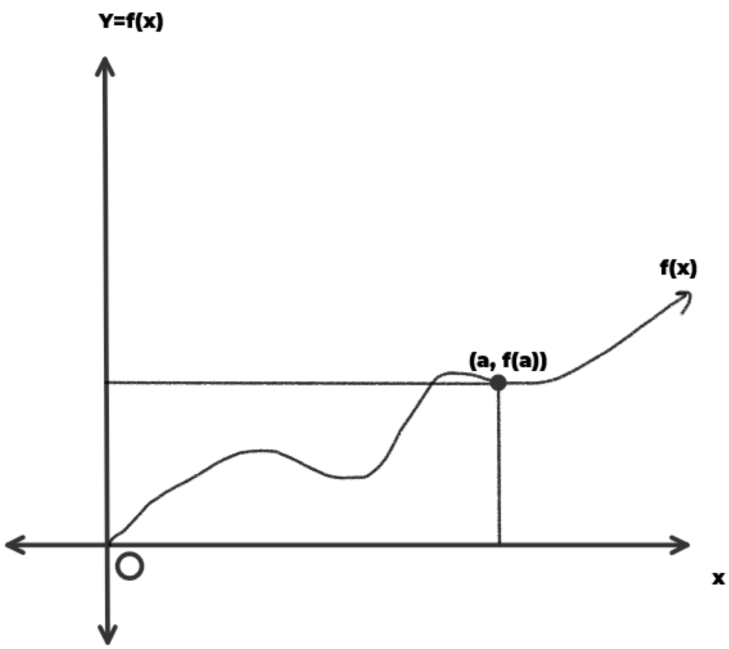
What Are Derivatives?

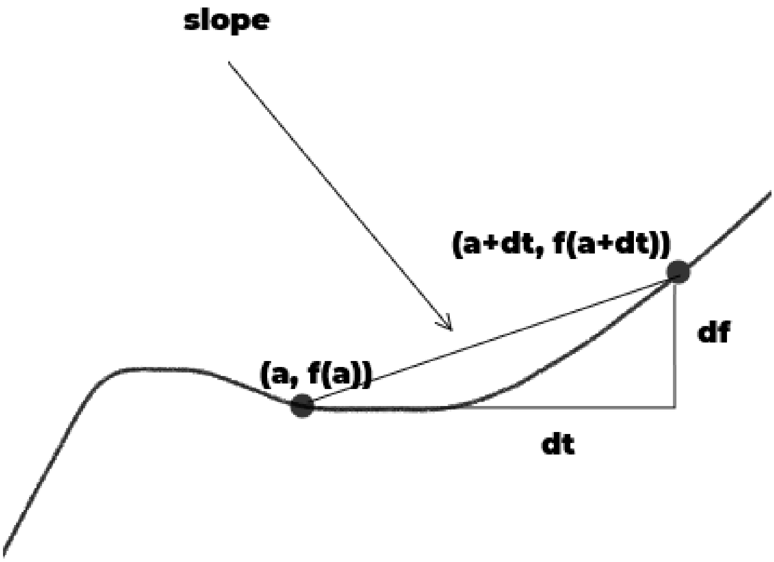
The derivative of a function is usually referred to as the instantaneous rate of change at a given point. This is achieved through finding the change in distance divided by the change in time, where that time instance becomes smaller and smaller. In other words, the slope between the two points used, the original point and another point across the function, reaches its true approximation.

The instantaneous rate of change being what occurs when the change in time becomes infinitesimally small. This presents a point of paradox as it approaches a length of time conceivable only as a length of zero units.

Now, to show this, imagine a graph with function $f(x)$ and an arbitrary point labeled $(a, f(a))$.



If we zoom in on the area by the arbitrary point and move along the function some unit of length, over a time horizontally measurable (our change in time) by dt and vertically measurable and labeled by df, which also functions as the change in distance, we can find our slope. This slope eventually becomes the derivative, as dt decreases in length towards zero, as referenced earlier.



Thus, in conclusion, the derivative of some function, at some point, can be described as:

$$\frac{df}{dt}(x) = \lim_{dt \to 0}\left[\frac{f(a+dt) - f(a)}{dt}\right]$$

Example:

$f(x) = 2x + 4x^3$

$$\lim_{dt \to 0} \frac{2(x + dt) + 4(x + dt)^3 - 2x - 4x^3}{dt}$$

$$\lim_{dt \to 0} \frac{2x + 2dt + 4(x^3 + 3x^2 dt + 3x dt^2 + dt^3) - 2x - 4x^3}{dt}$$

$$\lim_{dt \to 0} \frac{2dt + 4x^3 + 12x^2 dt + 12x dt^2 + 4dt^3 - 4x^3}{dt}$$

$$\lim_{dt \to 0} \frac{2dt + 12x^2 dt + 12x dt^2 + 4dt^3}{dt}$$

$$\lim_{dt \to 0} 2 + 12x^2 + 12x dt + 4dt^2$$

$\therefore 12x^2 + 2$