Integers Expressible as a Difference of Two Squares

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Given that the difference of two squares is formatted as $a^2 - b^2 = c$, find quantities such that $a^2 = (b + d)^2$ so that $a^2 - b^2 = d^2 + 2bd$, where $a, b, c, d \in \mathbb{Z}$

Case 1: If b and d are odd then they can be expressed as $\boldsymbol{b = 2e + 1}$ and $\boldsymbol{d = 2f + 1}$, where $\boldsymbol{e, f} \in \mathbb{Z}$.

$\therefore d^2 + 2bd$

$(2f + 1)^2 + 2(2e + 1)(2f + 1)$

$4f^2 + 4f + 1 + 2(4ef + 2e + 2f + 1)$

$4f^2 + 4f + 1 + 8ef + 4e + 4f + 2$

$4f^2 + 8f + 8ef + 4e + 3$

$2(2f^2 + 4f + 4ef + 2e + 1) + 1$

$2g + 1$, where $g = 2f^2 + 4f + 4ef + 2e + 1 \in \mathbb{Z}$

Thus, if b and d are odd then $c = 2g + 1$ which represents all odd integers. Thus, all odd integers can be expressed as a difference of two squares.

Case 2: If d is even and b is odd then they can be expressed as $\boldsymbol{d = 2h}$ ***and*** $\boldsymbol{b = 2i + 1}$ ***where*** $\boldsymbol{h, i \in \mathbb{Z}}$.

$\therefore d^2 + 2bd$

$(2h)^2 + 2(2h)(2i + 1)$

$4h^2 + 4(2hi + h)$

$4(h^2 + h + 2hi)$

$4j$, where $j = h^2 + h + 2hi \in \mathbb{Z}$

Thus, when d is even and b is odd then c is a multiple of 4.

Case 3: If b is even and d is odd then they can be expressed as $\boldsymbol{b = 2k}$ ***and*** $\boldsymbol{d = 2m + 1}$ ***where*** $\boldsymbol{k, m \in \mathbb{Z}}$.

$\therefore d^2 + 2bd$

$(2m + 1)^2 + 2(2k)(2m + 1)$

$4m^2 + 4m + 1 + 4(2mk + k)$

$2(2m^2 + 2m + 4mk + 2k) + 1$

$2n + 1$, where $n = 2m^2 + 2m + 4mk + 2k \in \mathbb{Z}$

Thus, when b is even and d is odd then c is an odd integer.

Case 4: If both b and d are even then they are expressible as $\boldsymbol{b = 2p}$ ***and*** $\boldsymbol{d = 2q}$***, where*** $\boldsymbol{p, q \in \mathbb{Z}}$***.***

$\therefore d^2 + 2bd$

$(2q)^2 + 2(2p)(2q)$

$4q^2 + 8pq$

$4(q^2 + 2pq)$

$4r$, where $r = q^2 + 2pq \in \mathbb{Z}$

Thus, when both b and d are even then c is a multiple of 4.

In conclusion, if c is expressed as an odd integer or a multiple of 4 it can be expressed as a difference of squares.